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ABSTRACT

This guide represents the final experimental version of a pilot project which was conducted in the United States between 1973 and 1976. The ideas and the manner of presentation are based on the works of Georges and Frederique Papy. They are recognized for having introduced colored arrow drawings ("papygrams") and models of our numeration system (the Papy "minicomputer") into the teaching of mathematics at the elementary and secondary level in Belgium. This program follows the "spiral approach." This guide reviews some basic numerical activities and introduces some new concepts. Topics included are composition, multiples and divisors, combinatorics, the minicomputer, decimals, multiplication, fractions and division. The standard minicomputer moves are reviewed. (Author/SA)

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Comprehensive School Mathematics Program
CEMREL, Inc.
Experimental Version
5-25501

CSMP Mathematics for the Intermediate Grades
Part III

Teacher's Guide

The World of Numbers

U.S. DEPARTMENT OF HEALTH,
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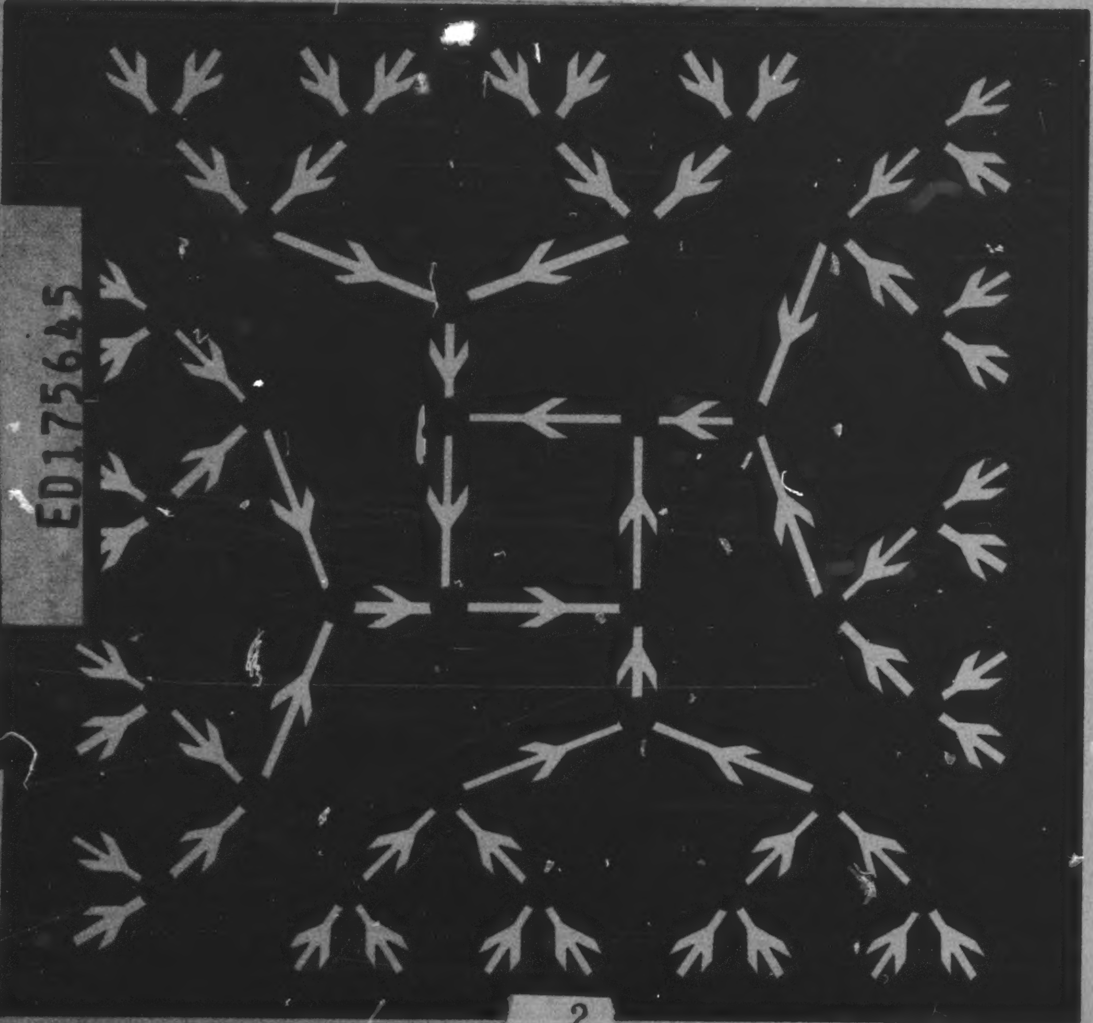

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Comprehensive School Mathematics Program

Mathematics
for the
Intermediate Grades
Part III

The World of Numbers

CEMREL, Inc.

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SUGGESTED SCHEDULE FOR N - LESSONS

WEEK	The World of Numbers	Materials
1	**1N** N1.1 Arrow Roads #1 p. 3	Teacher: Colored chalk Student: Pencil; paper; Worksheets N01, N02, N03, N04
2	**3N** N3.1 Hand-Calculator Sentences p. 65	T: Colored chalk S: Pencil; paper; Worksheets N17, N18, N19
3	**5N** N5.1 Hand-Calculator Relations with Decimals p. 170	T: Hand-calculator**, colored chalk S: Hand-calculator; pencil; paper; Worksheets N35, N36, N37, N38
4	**7N** N3.2 Spies and Bridges #1 p. 76	T: Colored chalk S: Pencil; paper; Worksheets N20, N21, N22, N23
5	**9N** N1.2 Arrow Roads #2 p. 11	T: Colored chalk S: Pencil; paper; Worksheets N05, N06, N07, N08
6	**11N** N7.2 Fractions #2 p. 250	T: Colored chalk S: Pencil; paper; Worksheets N62, N63, N64, N65
7	**13N** N5.3 Arrow Roads with Decimals p. 192	T: Colored chalk S: Colored pencils; pencil; paper; Worksheets N43, N44, N45, N46
8	**15N** N6.2 Multiplication #2 p. 227	T: Colored chalk S: Pencil; paper; Worksheets N54, N55, N56, N57
9	**17N** N4.2 Minicomputer Golf Preparation #2 p. 154	T: Minicomputer set; Weighted Checker set; colored chalk S: Pencil; paper; Worksheets N31, N32, N33, N34

** For all lessons listing hand-calculators as materials, see the General Introduction Section 2(f) on Role and Use of Hand-Calculators. While two hand-calculators are adequate for these lessons, the ideal situation would be for each student to have one. Encourage students to bring their own hand-calculators.

SUGGESTED SCHEDULE FOR N - LESSONS

The World of Numbers	Materials
2N N2.1 String Pictures p. 23	T: Colored chalk; Minicomputer set* S: Pencil; paper; Worksheets N09, N10, N11, N12
4N N4.1 Minicomputer Golf Preparation #1 p. 144	T: Minicomputer set; Weighted Checker set S: Hand-calculator (optional); pencil; paper; Worksheets N28, N29, N30
6N N6.1 Multiplication #1 p. 214	T: Colored chalk S: Colored pencils; pencil; paper; Worksheets N50, N51, N52, N53
8N N7.1 Fractions #1 p. 241	T: Colored chalk S: Colored pencils; pencil; paper; Worksheets N58, N59, N60, N61
10N N5.2 Decimals on the Number Line p. 183	T: Colored chalk S: Pencil; paper; Worksheets N39, N40, N41, N42
12N N8.1 Decimal Distance #1 p. 312	T: Colored chalk S: Colored pencils; pencil; paper
14N N3.3 Spies and Bridges #2 p. 90	T: Colored chalk S: Pencil; paper; Worksheet N24
16N N[illegible].1 Division #1 p. 332	T: Colored chalk S: Colored pencils; pencil; paper
18N N8.2 Decimal Distance #2 p. 322	T: Colored chalk S: Colored pencils; pencil; paper

*** A teacher's Minicomputer set consists of four demonstration Minicomputer boards and a sufficient number of demonstration Minicomputer checkers.**

SUGGESTED SCHEDULE FOR N - LESSONS

WEEK	The World of Numbers	Materials
10	**19N** N7.3 A New Name for One-Third p. 258	T: Minicomputer set (optional); colored pencils S: Pencil; paper
11	**21N** N7.4 Dangerous Roads p. 273	T: None S: Pencil; paper; Worksheets N66, N67
12	**23N** N2.2 Arrow Pictures for "Is a Positive Divisor of" p. 42	T: Colored chalk S: Pencil; paper
13	**25N** N4.3 Minicomputer Golf p. 163	T: Minicomputer set; Weighted Checker set; colored chalk S: Minicomputer set*; hand-calculator (optional)
14	**27N** N9.3 Division #3 p. 351	T: Colored chalk S: Colored pencils; pencil; paper; Worksheets N76, N77, N78, N79
15	**29N** N10.2 Who is Pif? p. 370	T: Hand-calculator; colored chalk; Minicomputer set; Weighted Checker set S: Hand-calculator; pencil; paper; Worksheets N82, N83, N84
16	**31N** N3.4 How many Permutations? p. 104	T: 8 small pieces of paper; colored chalk S: Pencil; paper
17	**33N** N3.5 Random Art #1 p. 118	T: 1 red and 1 blue marble; World of Numbers Poster #1; colored chalk S: Colored pencils; pencil; paper; Worksheets N25, N26, N27
18	**35N** N3.6 Random Art #2 p. 132	T: Colored chalk; World of Numbers Poster #1 S: Colored pencils; pencil; paper

* These materials are not supplied with the classroom set of materials for CSMP Mathematics for the Intermediate Grades, Part III. The student Minicomputer sets can be borrowed from a first, second, or third grade CSMP class, or 12 individual Minicomputer Sheets and 6 sheets of checkers can be purchased from CSMP.

SUGGESTED SCHEDULE FOR N - LESSONS

The World of Numbers	Materials
20N N9.2 Division #2 p. 342	T: Colored chalk S: Colored pencils; pencil; paper
22N N10.1 Who are Zig and Zag? p. 360	T: Colored chalk S: Pencil; paper; Worksheets N80, N81
24N N7.5 Guess My Rule p. 281	T: Colored chalk S: Pencil; paper; Worksheets N68, N69, N70
26N N2.3 Hand-Calculator Relations p. 51	T: Hand-calculator; colored chalk S: Hand-calculator; pencil; paper; Worksheets N13, N14, N15, N16
28N N7.6 A New Name for One-Half p. 291	T: Minicomputer set (optional); colored chalk S: Pencil; paper; Worksheet N71
30N N7.7 Fractions #3 p. 305	T: Colored chalk S: Pencil; paper; Worksheets N72, N73, N74, N75
32N N5.4 Decimals on the Minicomputer p. 199	T: Minicomputer set; hand-calculator S: Hand-calculator; pencil; paper; Worksheets N47, N48, N49
34N N10.3 Who is Jack? p. 378	T: Hand-calculator, colored chalk; Minicomputer set; Weighted Checker set S: Hand-calculator; pencil; paper
ADJUSTMENT DAY	

SUGGESTED CHRONOLOGICAL ORDER OF N - LESSONS

Lesson One	N1.1 - Arrow Roads #1	3
Lesson Two	N2.1 - String Pictures	23
Lesson Three	N3.1 - Hand-Calculator Sentences . . .	65
Lesson Four	N4.1 - Minicomputer Golf Preparation #1	144
Lesson Five	N5.1 - Hand-Calculator Relations with Decimals	170
Lesson Six	N6.1 - Multiplication #1	214
Lesson Seven	N3.2 - Spies and Bridges #1.	76
Lesson Eight	N7.1 - Fractions #1	241
Lesson Nine	N1.2 - Arrow Roads #2	11
Lesson Ten	N5.2 - Decimals on the Number Line . . .	183
Lesson Eleven	N7.2 - Fractions #2	250
Lesson Twelve	N8.1 - Decimal Distance #1.	312
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Lesson Fifteen	N6.2 - Multiplication #2	227
Lesson Sixteen	N9.1 - Division #1	332
Lesson Seventeen	N4.2 - Minicomputer Golf Preparation #2	154
Lesson Eighteen	N8.2 - Decimal Distance #2.	322
Lesson Nineteen	N7.3 - A New Name for One-Third . . .	258
Lesson Twenty	N9.2 - Division #2	342
Lesson Twenty-one	N7.4 - Dangerous Roads	273
Lesson Twenty-two	N10.1 - Who are Zig and Zag? . . .	360
Lesson Twenty-three	N2.2 - Arrow Pictures for "Is a Positive Divisor of"	42
Lesson Twenty-four	N7.5 - Guess My Rule	281
Lesson Twenty-five	N4.3 - Minicomputer Golf	163
Lesson Twenty-six	N2.3 - Hand-Calculator Relations . . .	51
Lesson Twenty-seven	N9.3 - Division #3	351
Lesson Twenty-eight	N7.6 - A New Name for One-Half . . .	291
Lesson Twenty-nine	N10.2 - Who is Pif?	370

Lesson Thirty N7.7 - Fractions #3. 305
Lesson Thirty-one N3.4 - How Many Permutations? . . . 104
Lesson Thirty-two N5.4 - Decimals on the Minicomputer . . 199
Lesson Thirty-three N3.5 - Random Art #1 118
Lesson Thirty-four N10.3 - Who is Jack? 378
Lesson Thirty-five N3.6 - Random Art #2 132

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THE WORLD OF NUMBERS

INTRODUCTION

By this time, your students have had a rich variety of experiences with various kinds of numbers and operations and relations on them. These include encountering both positive and negative integers, decimal numbers, some rational numbers, numerical functions (e.g., 2×; +3; ÷10; −5), order relations (e.g., < and >), and the notions of multiples and positive divisors of a given number. They have been introduced to paper and pencil algorithms for addition, subtraction, and multiplication of whole numbers and decimal numbers, and they have had extensive experiences using systematic methods for division and for multiplication and addition of rational numbers (fractions) in preparation for algorithms.

In the present strand, these previous numerical experiences will be revisited, extended, and deepend. New situations will be presented, including the introduction of several fascinating extensions and modifications of numerical games: Minicomputer Golf (Chapter 4); Guess My Rule (Section 7.5); Binary and other number base abaci (Section 3.2, Section 3.3, Section 7.3, and Section 7.6); Detective stories (Chapter 10); and new numerical games: Decimal Distance (Chapter 8) and Hand-Calculator Sentences (Section 3.1). Old and new concepts will be explored in the context of these situations.

The first paragraph above, describing some of your students' previous numerical experiences, simply catalogues some of the realms of the world of numbers your students have visited. It does not indicate the travel routes nor the vehicles used to travel them.

The CSMP curriculum stresses the unity and continuity of the growth of mathematical ideas and concepts. This is reflected in numerous ways in this strand, as in the others. For example, CSMP employs the spiral approach, not pausing for mastery of each lesson but rather allowing the student to encounter the elements of each content strand in different situations, interspersed with other strands throughout the year. It is important to recognize this approach consciously. If you strive for mastery of each single lesson, you will find yourself involved in a great deal of redundancy as the year progresses.

Further, CSMP presents the content in a situational framework. That is, a "pedagogy of situations" is employed which engages students in rich problem solving activities. These offer the opportunity both to develop necessary numerical skills and to gain deeper understanding of mathematical ideas and concepts in the world of numbers. At the same time the situations presented encourage the student to develop patterns of logical thinking and strategies for attacking problems.

Perhaps the most important embodiments of the CSMP approach are the three nonverbal languages used throughout its elementary program: the Papy Minicomputer, the language of arrows, and the language of strings. These are the vehicles which make it possible to develop the situations in which the content is presented to the student. Further, as the students grow in the CSMP program, these languages become basic investigatory tools available to explore the content.

It is hard to overstate the value of developing tools which aid in attacking problems as well as in representing situations; tools which are not confined to one area of mathematical content or to one level of the development of content. In the usual elementary curriculum, students must continually learn new language to express each new piece of mathematics that is studied. Armed with the universally applicable languages of the CSMP program, the students grow more and more familiar with the syntax of these languages and are freed to explore new content directly in terms of these same languages rather than being forced to think of each new mathematical idea as tied to a certain new language. This is definitely not to say that the CSMP students do not learn the usual descriptive language of mathematics. However, in the CSMP program the usual descriptive language is not the vehicle for learning, but only for describing the ideas.

For specific examples of the situations and the ways the CSMP languages are used in this strand, we refer you to the short introductory remarks preceding each chapter and, in particular, to the lessons themselves.

A WORD ABOUT STANDARD ALGORITHMS

We mentioned above that the CSMP program seeks to develop the necessary skills in the numerical domain as well as deeper understanding of the underlying mathematical ideas. We are fully in agreement with the thesis that along with the growth of understanding in the world of numbers, there must be a concommitant growth of familiarity and facility with numbers and the operations and relations on them. But facility must not be confused with understanding. They are partners in the growth of mathematical maturity, but it is extremely important that neither be sacrificed for the other. Rather, a balanced growth in each must be maintained.

In the usual elementary curriculum, paper and pencil algorithms for the four basic operations provide a prime opportunity for the abuse of the position stated above. It is undeniably important that a student eventually learns mechanical algorithms for the basic operations. However, if these algorithms are presented prematurely they stunt the student's ability to develop alternative algorithms, to do mental arithmetic, to estimate, and so on.

For example, rather than employing the standard algorithm to determine the standard name for 1,791 ÷ 9, it may be easier and more efficient to note that 200 x 9 = 1,800, so that 199 x 9 = 1,791 and 1,791 ÷ 9 = 199. Indeed, built into this way of approaching the problem is a relatively good estimate (1,800 ÷ 9) of the number involved.

If we insist on a mechanistic response to problems such as the one above (i.e., insist on the use of the paper and pencil algorithm), we may not only encourage inefficiency in many situations but also inhibit the development of the flexibility necessary for problem solving. On the other hand, if we present to students a rich array of situations in which they interact with numbers and their operations and let the standard algorithms grow naturally as just one facet of the students' experience, we obtain two important results. The students will gain the necessary facility with standard algorithmic procedures and at the same time retain the openness required to respond creatively to new situations in the world of numbers.

THE STANDARD MINICOMPUTER PLAYS

Children who start the CSMP curriculum in Kindergarten or Grade One meet the Minicomputer for the first time just when they are starting to learn about numbers and the various operations on numbers. In such a context, the Minicomputer is used as a support for the positional system of numeration and as a calculating tool that helps to strengthen the first ideas of calculation and the learning of the "basic facts". To this end, heavy emphasis is laid on the "standard" representation of numbers on the Minicomputer (i.e., the representation that uses the fewest (positive) checkers possible digit by digit) and on the "standard" plays on the Minicomputer (these are described in detail below), because these provide the most efficient means to the end of teaching the children about elementary arithmetic.

Each of the "standard" Minicomputer plays leaves unchanged the number represented on the Minicomputer. Later, in the CSMP Upper Primary Grades Program (second and third grades), the children meet moves which either increase or decrease the number represented on the Minicomputer. Your students are already quite familiar with both the "standard" and "non-standard" Minicomputer plays. Many of them have probably developed shortcuts and many recognize the answer to a problem without putting it in "standard form".

Nonetheless, it is important for the students to realize that there is a mechanical method for performing all the "standard" plays. It requires no intuition and and could be done by a suitably designed machine. For this reason the "standard" plays become known in the Intermediate Grades Program as "robot" plays. In cases where the mechanical nature of the Minicomputer is to be emphasized the students are asked to make "robot" plays and are limited to the following:

1 + 1 = 2 10 + 10 = 20 100 + 100 = 200 1,000 + 1,000 = 2,000	2 = 1 + 1 20 = 10 + 10 200 = 100 + 100 2,000 = 1,000 + 1,000
two checkers on the white square of a board = one checker on the red square of the same board	one checker on the red square of a board = two checkers on the white square of the same board

2 + 2 = 4 20 + 20 = 40 200 + 200 = 400 2,000 + 2,000 = 4,000	4 = 2 + 2 40 = 20 + 20 400 = 200 + 200 4,000 = 2,000 + 2,000
two checkers on the red square of a board = one checker on the purple square of the same board	one checker on the purple square of a board = two checkers on the red square of the same board

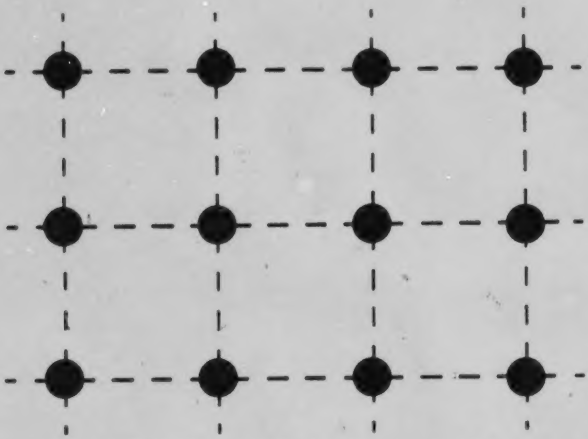
4 + 4 = 8 40 + 40 = 80 400 + 400 = 800 4,000 + 4,000 = 8,000	8 = 4 + 4 80 = 40 + 40 800 = 400 + 400 8,000 = 4,000 + 4,000
two checkers on the purple square of a board = one checker on the brown square of the same board	one checker on the brown square of a board = two checkers on the purple square of the same board

8 + 2 = 10 80 + 20 = 100 800 + 200 = 1,000 8,000 + 2,000 = 10,000	10 = 8 + 2 100 = 80 + 20 1,000 = 800 + 200 10,000 = 8,000 + 2,000
one checker on each of the brown and red squares of a board = one checker on the white square of the next board to the left	one checker on the white square of a board = one checker on each of the brown and red squares of the next board to the right[*]

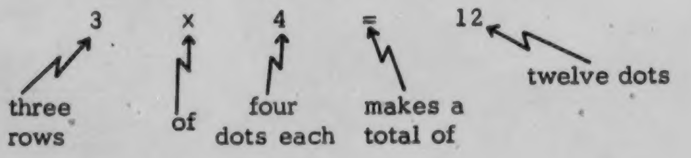
[*] This provides the link that motivates the introduction of the decimal numbers. In particular, we have that one checker on the white square of the ones' board equals one checker on each of the brown and red squares of a special board (the tenths' board - or as it is often referred to in the lessons, the dimes' board) located to the right of the ones' board and separated from it by a special barrier.

NOTE ON THE ORDER OF MULTIPLICATION

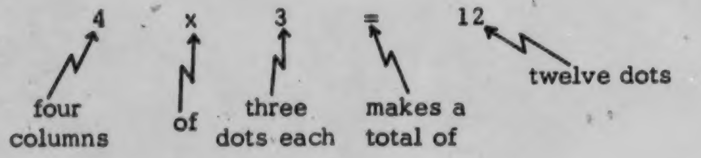
Should the function "multiply by 2" be denoted by "2 ×" or "× 2"? Does it matter? The answer to this last question is "yes and no". It depends on your viewpoint and on the context in which you want to use it. In order to appreciate this fact a little better, let us go back to the very beginning: multiplication of nonnegative whole numbers. How does the following array of 12 dots appear to you?



If you see three rows of four (the red dashed lines), this fact can be recorded as:


3
×
4
=
12
three rows
of
four dots each
makes a total of
twelve dots

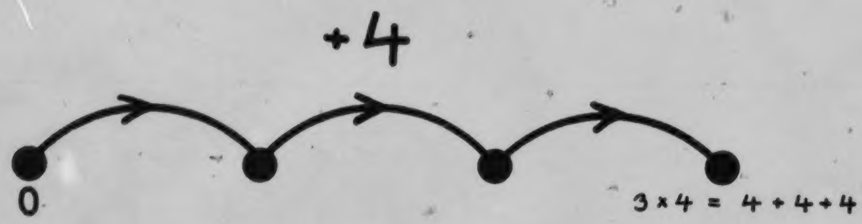
On the other hand, if you see four columns of three (the blue dashed lines), this fact can be recorded as:


4
×
3
=
12
four columns
of
three dots each
makes a total of
twelve dots

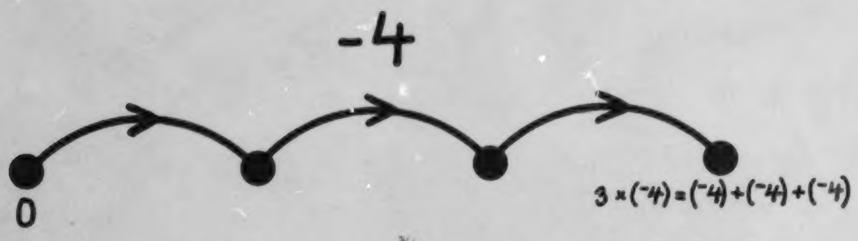
In that they describe two different perceptions, "4 × 3" and "3 × 4" are obviously very different. But, equally obvious is that "4 × 3" and "3 × 4" are both names for the same number, namely 12. So, in this case, it makes no difference if we denote "multiply by 3" by "×3" or "3×". This is true for any nonnegative integer, n: "×n" and "n×" may be used indifferently to denote "multiply by n", provided the number you are multiplying by n is also a nonnegative integer.

Problems do not arise until negative integers and (non-integer) decimal numbers put in their appearance; the reason being that it is a bit more difficult to find a model for multiplication when negative integers and (non-integer) decimal numbers are around. What would it mean, for example, to have an array of dots with six rows each containing negative two dots, or, even worse, negative three rows each containing negative five dots? What about 3.72 rows each containing seven dots? Fortunately, it is possible to find a model which allows the calculation of at least some products involving negative integers or (non-integer) decimal numbers; but, unfortunately, we lose the advantage of the array-of-dots model which makes it evident that the order of multiplication does not influence the answer. We call upon the language of arrows

- 3 × 4 is the number obtained if you start from 0 and use three + 4 arrows.



- $3 \times (^{-}4)$ is the number obtained if you start from 0 and use three $^{-}4$ arrows.



This model has the advantage that it not only "pictures" the product, but it also provides a means of calculating that product: just label all the dots until you reach the one in which you are interested. The disadvantage is that it doesn't make any sense to speak of a negative number of arrows starting from 0. Thus, strictly speaking, your children have only been introduced to products in which the first number is a whole number (the second number can be either positive or negative or zero and can even be a (non-integer) decimal number). This has implications for notation.

- As noted above, if only nonnegative integers are involved, you can use "2 ×" or "× 2" as the fancy takes you (we have scattered both of these notations throughout this semester's curriculum).

- If a negative integer (say, $^{-}4$) is being multiplied by 2 (say), then, strictly speaking, you should denote this "$2 \times (^{-}4)$" and not "$^{-}4 \times 2$". This is because "$2 \times (^{-}4)$" carries the sense of "$(^{-}4) + (^{-}4)$", whereas "$^{-}4 \times 2$" has no natural connotation at this stage. In other words, strictly speaking, you should use "2 ×" when negative integers and (non-integer) decimal numbers are involved, not "× 2". We have observed this rule in the Teacher's Guide for this semester.

We have emphasized the phrase "strictly speaking" above because this is not something you should make into a big issue with your children. In fact, it would be best to avoid all mention of this subject in your classroom if at all possible. If any of your children denote "times" functions one way when strictly speaking they should do it the other way, simply accept it without comment. There could be a variety of reasons why some of your children might do this; among them are:

(i) to conform to a pattern: all the other types of functions are usually written with the operation symbol on the left (- 2; ÷ 3; + 6).

(ii) to generalize the results from the case of nonnegative integers: it is perfectly natural to believe that the order of multiplication makes no difference in the case of any integers.

Our concern in this note has been to make you aware of some of the underpinnings of this issue so that, in the very unlikely event that a problem arises in your class, you will have the necessary background information to understand why it might have occurred, thus perhaps pointing the way to a means of getting out of the difficulty.

Of course, you really have nothing to worry about; if your children infringe on any of the above "rules", what they do will not be wrong (because, after all, it really doesn't make any difference in which order you multiply two integers). It is just that, at this stage, what they do will be unfounded.

A WORD ON THE USE OF HAND-CALCULATORS[*]

The hand-calculator provides another powerful tool for the investigation of problems and situations in the world of numbers. Appropriately, this tool is a natural extension of the Papy Minicomputer, since its internal structure is an electronic simulation of the Minicomputer. This analogy need not be consciously acknowledged by you or by a student, but it does serve to indicate that the hand-calculator is indeed a natural tool in the CSMP program.

At least two hand-calculators need to be available for Part III of the CSMP Intermediate Grades Program. They are not included in your complete classroom set; however, they may be purchased from CSMP as separate items. There are extensive opportunities for the use of hand-calculators in the lessons of Part III, especially in Chapters 2, 3, and 5. In addition, the supplementary material for for the slow learner, "Math Play Therapy", relies heavily on the use of hand-calculators.

We are aware that it is unrealistic, indeed premature at this time, to develop extensive classroom materials which require that every student in the class have a hand-calculator. We are firmly convinced through our own experiences with children using hand-calculators that the hand-calculator is a potent pedagogical tool when used properly. The time is not far off when its presence in the classroom will be as commonplace as the pencil. For the present, CSMP will continue to investigate the use of the hand-calculator in the classroom, particularly with slow learners, and will develop situations designed for its use.

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[*] See General Introduction, Section 2 (f) for a more detailed discussion on hand-calculators.

We strongly suggest that for the lessons mentioned above you request your students who have hand-calculators to bring them to class. Even though only two hand-calculators are needed for these lessons, the pedagogical situation would be much improved if as many students as possible had access to a calculator during these lessons.

1

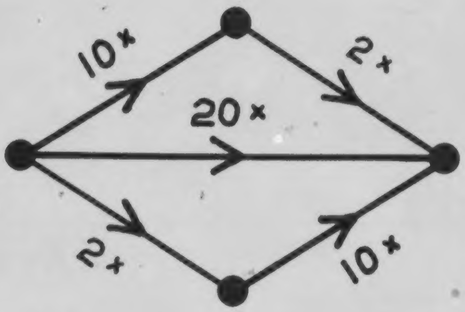
COMPOSITION

We continue to call upon the language of arrows to support our thinking and to record information as we further explore the World of Numbers.

The familiar numerical functions, such as $+3$, -7, $4\times$, $\div 5$, are in a sense basic to the arithmetic of whole numbers. By means of composition, functions such as $\frac{2}{3}\times$ (that is, the composite of $\div 3$ and $2\times$) are obtained; functions such as these are essential for the arithmetic of rational numbers.

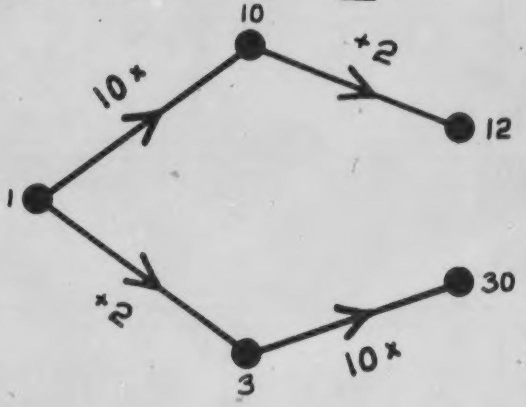
In this chapter, composition of numerical functions is reviewed in various settings. In particular, attention is again focused on the distinction between composites of, for example, $+2$ followed by $10\times$ and $10\times$ followed by $+2$. Notice that although

$$(10\times) \circ (2\times) = (2\times) \circ (10\times),$$



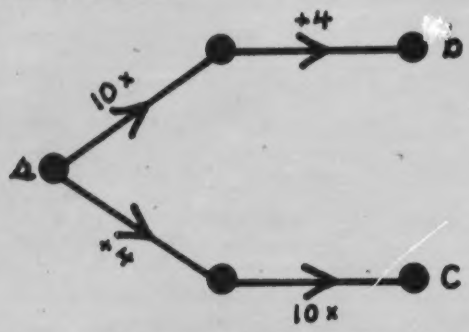
where here both functions are of the form "$\square\times$", this is not the case where one function is of the form "$\square\times$", and the other is of the form "$+\square$"; for example,

$$(10\times) \circ (+2) \neq (+2) \circ (10\times).$$



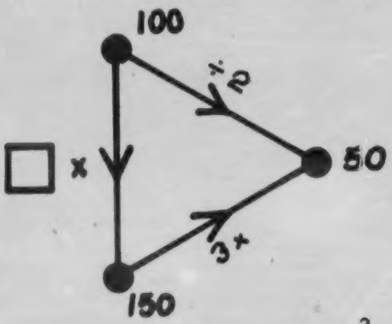
CAPSULE LESSON SUMMARIES

Section 1.1: Label the dots in arrow roads involving arrows for addition, subtraction, multiplication, and division functions; then draw arrows for certain compositions of these functions and identify the composite functions. In an arrow picture, such as



determine the numbers at b and c given a number at s, and establish that the difference between the numbers at c and b will be the same for each number at s. (Worksheets N01, N02, N03, N04)

Section 1.2: Label the dots and the arrows in an arrow road, where the arrows to be identifed are of the form "☐ x". Identify some of the arrows by using detours; for example,

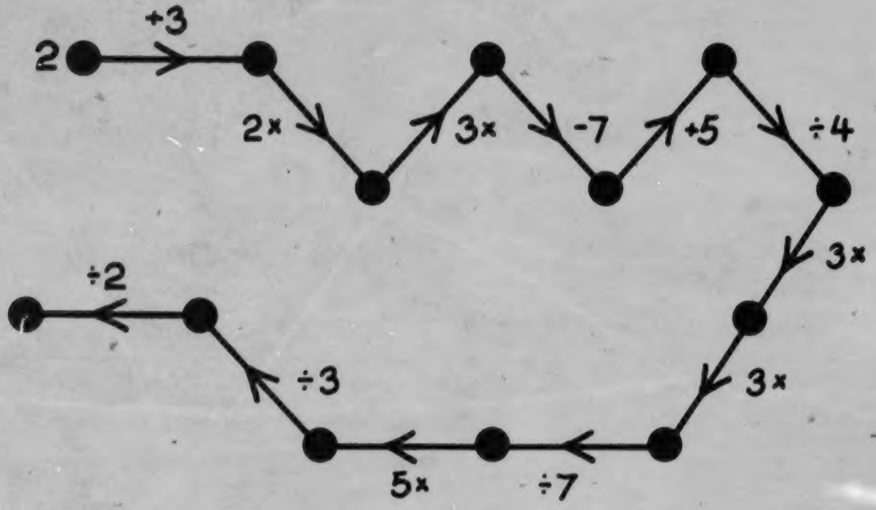


so that the red arrow is for $\frac{3}{2}$ x. Continue the latter activity of Section 1.1. In such an arrow picture, determine which dot is for the largest number; then, knowing this number, find the numbers at the other dots. (Worksheets N05, N06, N07, N08)

1.1 Arrow Roads #1

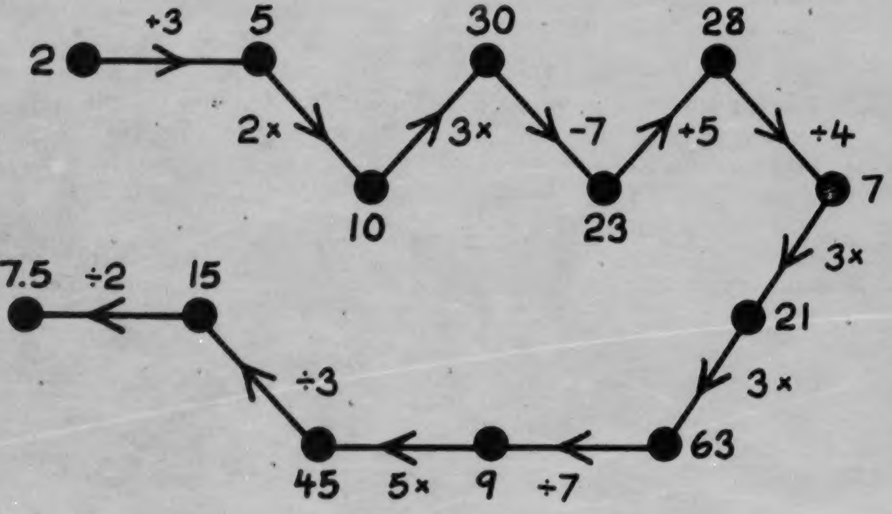
Exercise 1

Progressively draw this arrow road on the board.


2
+3
2×
3×
-7
+5
÷4
3×
3×
÷7
5×
÷3
÷2

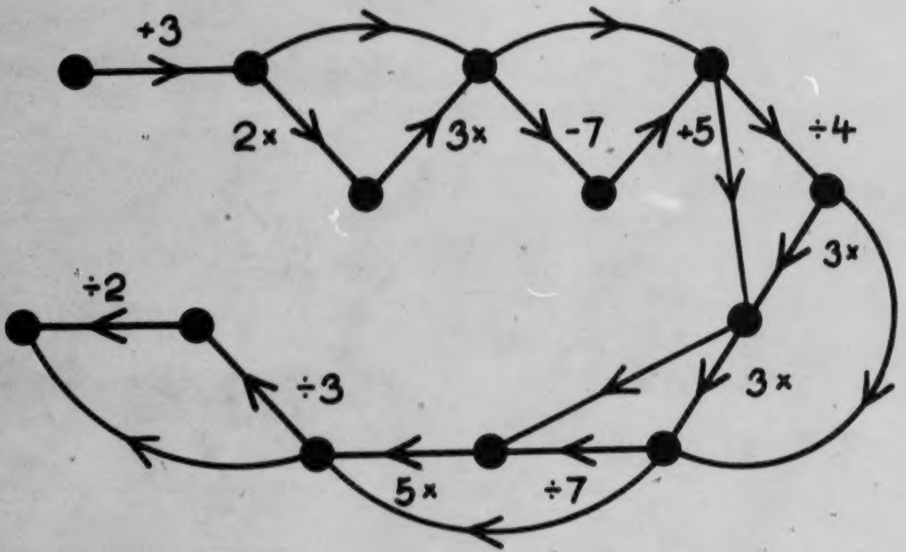
Have students label the dots. Encourage the students to make the calculations mentally.

A completely labeled picture is shown below.


2
+3
5
2×
10
3×
30
-7
23
+5
28
÷4
7
3×
21
3×
63
÷7
9
5×
45
÷3
15
÷2
7.5

1N

Erase the labels for the dots but leave the arrow labels. Add these blue arrows to your picture.



Trace the 2 x arrow followed by the first 3 x arrow. Then trace the blue arrow beginning at the same dot as the 2 x arrow.

T: This blue arrow is for 2 x followed by 3 x . What could this blue arrow be for?

S: 6 x .

Label the arrow "6 x" and trace the next blue arrow.

T: This blue arrow is for - 7 followed by + 5. What could this blue arrow be for?

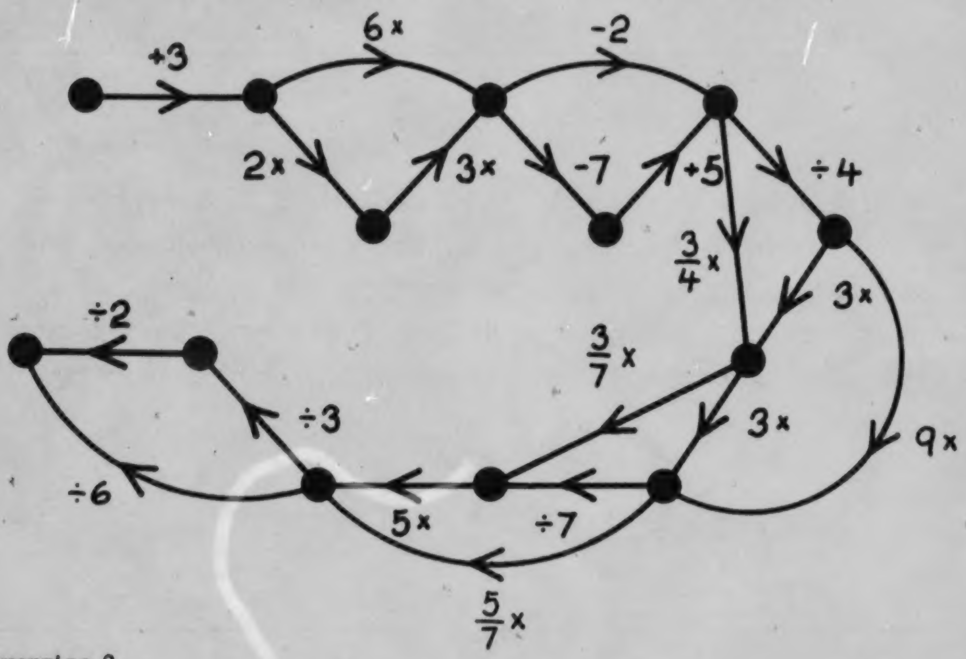
S: - 2.

Label the arrow "- 2" and continue.

T: ÷ 4 followed by 3 x . What could this blue arrow be for? Think a minute and then write it on your paper.

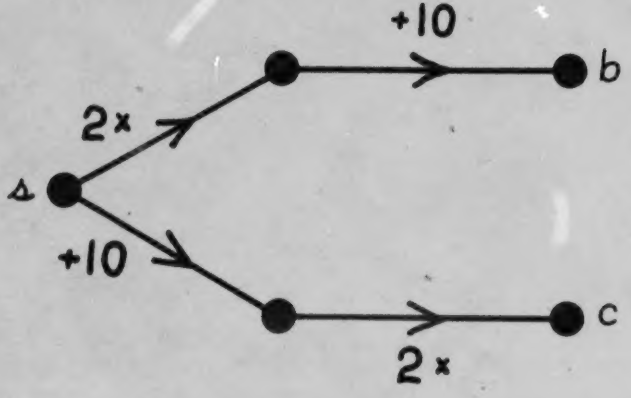
Let the students have a few minutes to consider this problem. If anyone has difficulty with this problem, remind them that ÷ 4 followed by 3 × is another way of expressing $\frac{3}{4}$ ×.

Continue in the same manner to label the remaining blue arrows. An arrow picture with correct labels on all the blue arrows is shown below.



Exercise 2

Draw this arrow picture on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

1N

T: All the dots in this picture are for positive numbers. (Point to b and c.) Which of these two numbers is larger?

S: The lower number (c) is larger.

T: Why?

S: If you start with 5 (at s), the upper number (b) is $(2 \times 5) + 10 = 20$ and the lower number (c) is $(5 + 10) \times 2 = 30$.

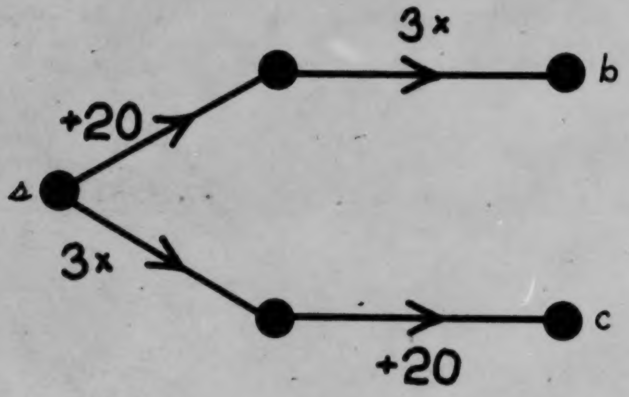
Select several more starting numbers (always positive). Each time label s and have the students label the other dots. Below are some suggested starting numbers for s with the corresponding ending numbers (b and c). Feel free to adjust the starting numbers to the level of your class. Encourage the students to make some observations about the relative sizes of the numbers at b and c.

s	b	c
7	24	34
11	32	42
3.2	16.4	26.4
8.5	27	37

T: Each time we started with a number here (point to s), the number we calculated to be here (point to c) was 10 greater than the numbers here (point to b). Do you think, in this arrow picture, that the lower number will always be 10 greater than the upper number? (Yes) Why?

S: In the lower part of the picture, 10 is added to a number and then the sum is doubled giving double the number plus 20. In the upper part of the picture, a number is doubled and then 10 is added to the result. So the lower number is 10 more than the upper number.

Erase the board and draw this new arrow picture.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: Again, the dots are for positive numbers. (Point to b and c.) Which number is larger?

S: The upper one (at b).

T: Why?

Accept any reasonable explanation even if it is as incomplete as the response below. The next questions call attention to the need for further clarification.

S: Because in the upper part you add 20 and then multiply the result by 3. You are multiplying a larger number by 3 for the upper part than you are for the lower part.

T: How much larger will this number (point to b) be than this number (point to c)?

S: 40 larger.

T: Why 40 larger?

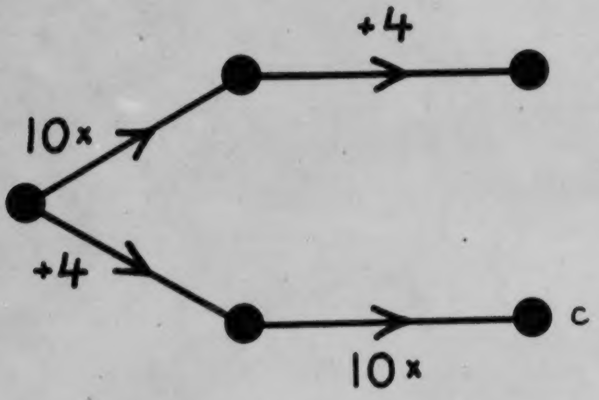
Along the upper path, 20 is added to the number at s and the sum is multiplied by 3. So the number at b is 3 times the number at s plus 60. Along the lower path, the number at s is multiplied by 3 and 20 is added to the product. So the number at c is 3 times the number at s plus 20. The number at b is 40 greater than the number at c.

Accept any explanation that is reasonable. It is not necessary to force the above explanation.

If some students are not convinced by the arguments, label the dot at s (for example, "10") and calculate the numbers at b and c. This should be repeated a few times.

Exercise 3

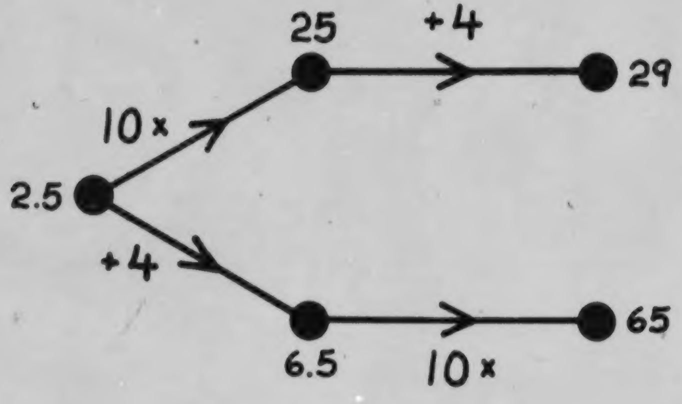
Draw this arrow picture on the board.



Do not write the letter on the board. It is here just to make the description of the lesson easier to follow.

T: All of the dots in this picture are for positive numbers and 65 is the largest. Copy this picture on your paper and circle the dot for 65.

Observe the students' pictures. After many students have finished the problem, have someone come to the board and point to the dot for 65 (c in the previous picture). Once 65 has been correctly placed, ask the students to label the remaining dots in their copy of the arrow picture. Then label the dots in your picture on the board collectively. The completed arrow picture is drawn below.

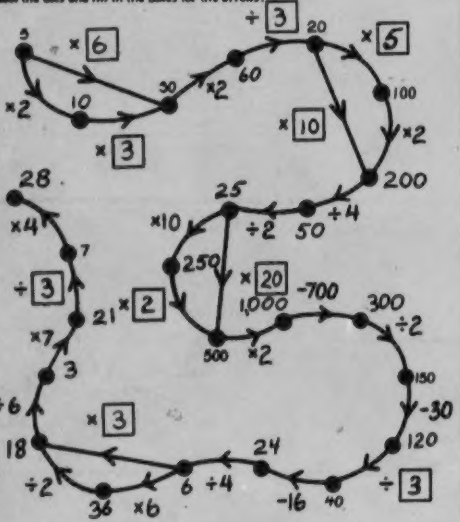


Worksheets N01, N02, N03, and N04 are available.

Name ______________

N 01 *

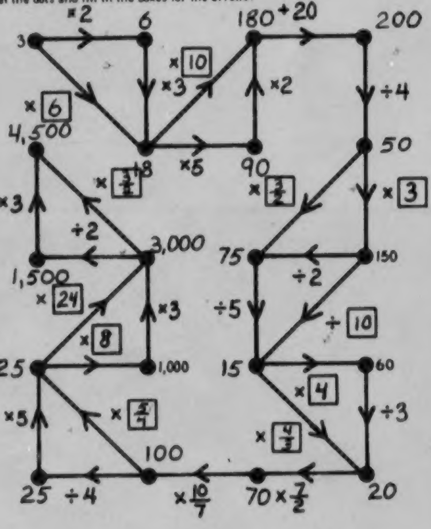
Label the dots and fill in the boxes for the arrows.


5
×6
×2
10
×3
30
×2
60
÷3
20
×5
100
×10
×2
200
÷4
50
÷2
25
×10
250
×20
×2
500
×2
1,000
-700
300
÷2
150
-30
120
÷3
40
-16
24
÷4
6
×3
×6
36
÷2
18
÷6
3
×7
21
÷3
7
×4
28

Name ______________

N 02 **

Label the dots and fill in the boxes for the arrows.

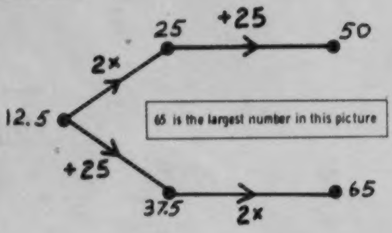

×2
3
6
×10
×3
180
+20
200
×2
÷4
×6
4,500
18
×5
90
50
×3/2
×3/2
×3
×3
÷2
3,000
75
150
1,500
÷2
×24
×3
÷5
÷10
×8
125
1,000
15
60
×4
×5
×5/4
×4/3
÷3
100
25
÷4
×10/7
70
×7/2
20

Name ______________

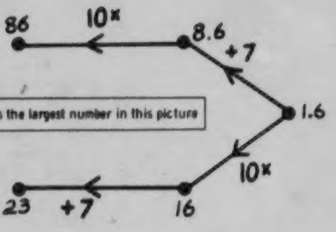
N 03 ***

All the dots are for positive numbers.

Label all the dots.


25
+25
50
2×
12.5
+25
37.5
2×
65

65 is the largest number in this picture

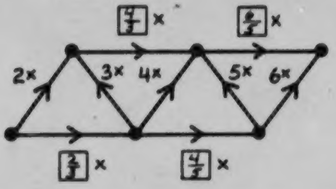

86
10×
8.6
+7
1.6
10×
23
+7
16

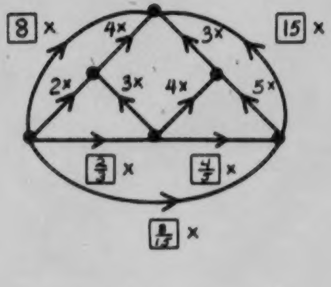
86 is the largest number in this picture

Name ______________

N 04 ****

Fill in the box for each blue arrow.

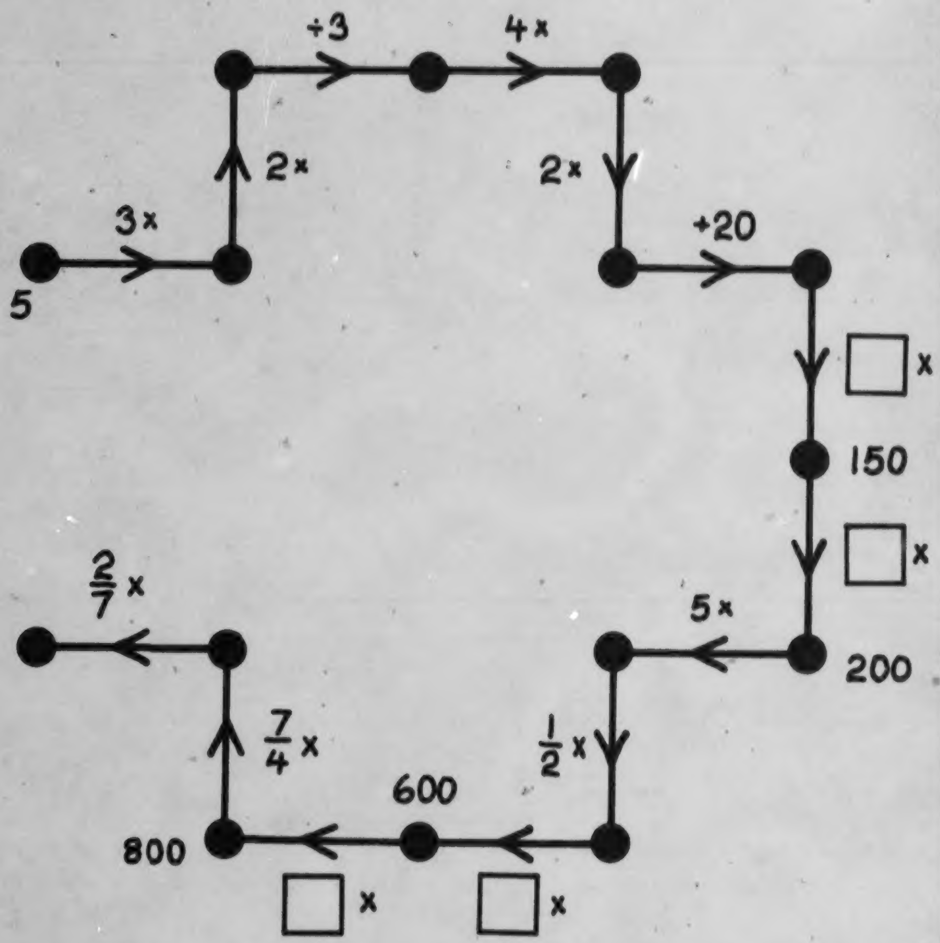

4/3 ×
6/5 ×
2×
3×
4×
5×
6×
2/3 ×
4/5 ×


8 ×
4×
3×
15 ×
2×
3×
4×
5×
2/3 ×
4/5 ×
8/15 ×

1.2 Arrow Roads #2

Exercise 1

Slowly and progressively draw this arrow road on the board.

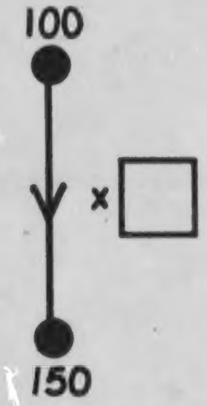

÷3
4x
2x
2x
3x
+20
5
☐x
150
☐x
$\frac{2}{7}$x
5x
200
$\frac{7}{4}$x
$\frac{1}{2}$x
600
800
☐x
☐x

T: As I draw this arrow road, you will help me label the dots and the arrows.

Have a student label a dot after you draw it or label an arrow ending at a dot that you have labeled. The first six dots should be labeled "15", "30", "10", "40", "80", and "100", respectively.

9N

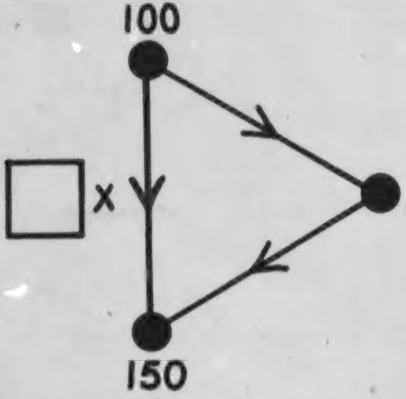
Labeling arrows is more challenging for most students. Suppose the students are trying to label the arrow that starts at 100 and ends at 150.



T: Label this arrow "☐x" (read: some number times).

S: We need to take a detour.

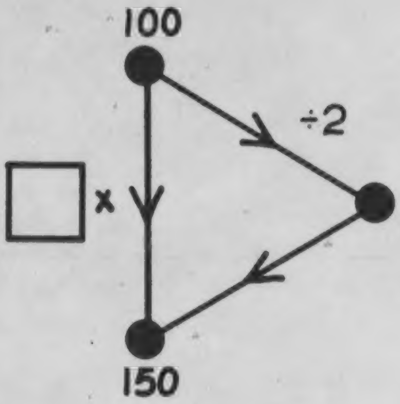
If this is not suggested, do it yourself. Draw these blue arrows and suggest that it might help to look at this detour.



T: We can label one of the blue arrows "☐x" and the other "÷☐" (read: divided by some number). What could the blue arrows be for?

S: One blue arrow could be for ÷ 2.

Label the first arrow "÷ 2".



T: What is 100 ÷ 2?

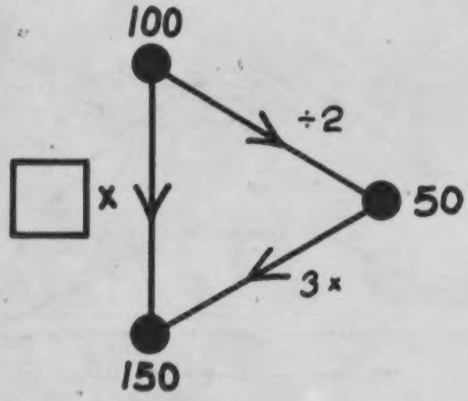
S: 50.

Label the dot "50".

T: What could the other blue arrow be for?

S: 3 ×.

Label the arrow "3 ×".


100
÷2
x
50
3 ×
150

T: Now that we have taken a detour, what could the red arrow be for?

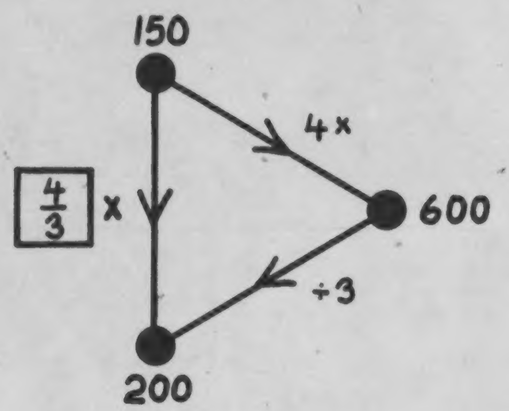
9N

S: $\frac{3}{2}$ x.

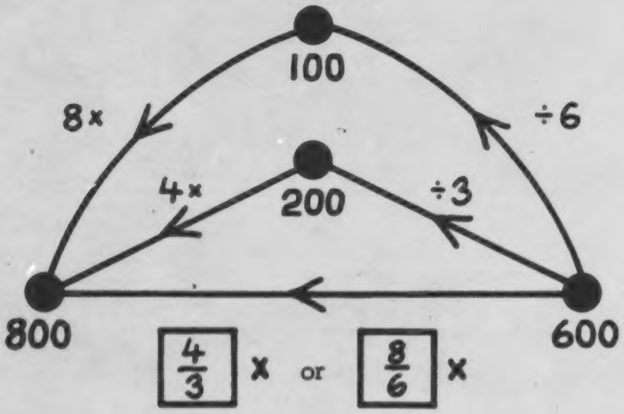
Label the red arrow "$\frac{3}{2}$ x". Draw the next arrow starting at 150 and ending at 200.

T: What could this red arrow be for?

Some students will probably suggest that a detour is necessary. One helpful detour is drawn below. Lead the discussion in a manner similar to the above.

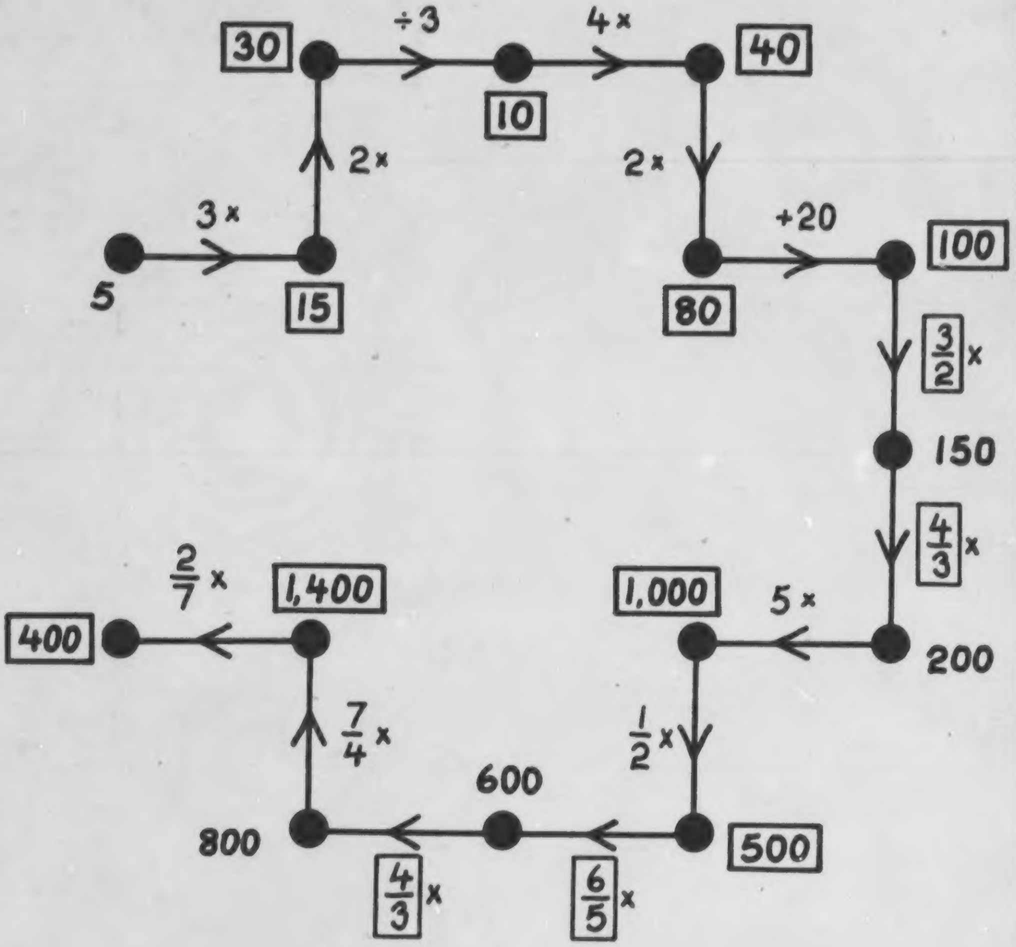

150
4 x
$\boxed{\frac{4}{3}}$ x
600
÷3
200

It is possible that students will suggest different detours. This situation could occur at any time but seems, at least in our experience, most likely to happen when one is labeling the arrow starting at 600 and ending at 800. In our development classes, these detours were suggested.


100
8 x
÷6
4 x
200
÷3
800
600
$\boxed{\frac{4}{3}}$ x or $\boxed{\frac{8}{6}}$ x

Keep in mind that at any time several useful detours may be suggested, and whenever this occurs note that the composite functions are equal. In the last example, the following are all names for the same function: "$\frac{4}{3}$ x", "$\frac{8}{6}$ x", "$\frac{12}{9}$ x", "$\frac{16}{12}$ x", ...

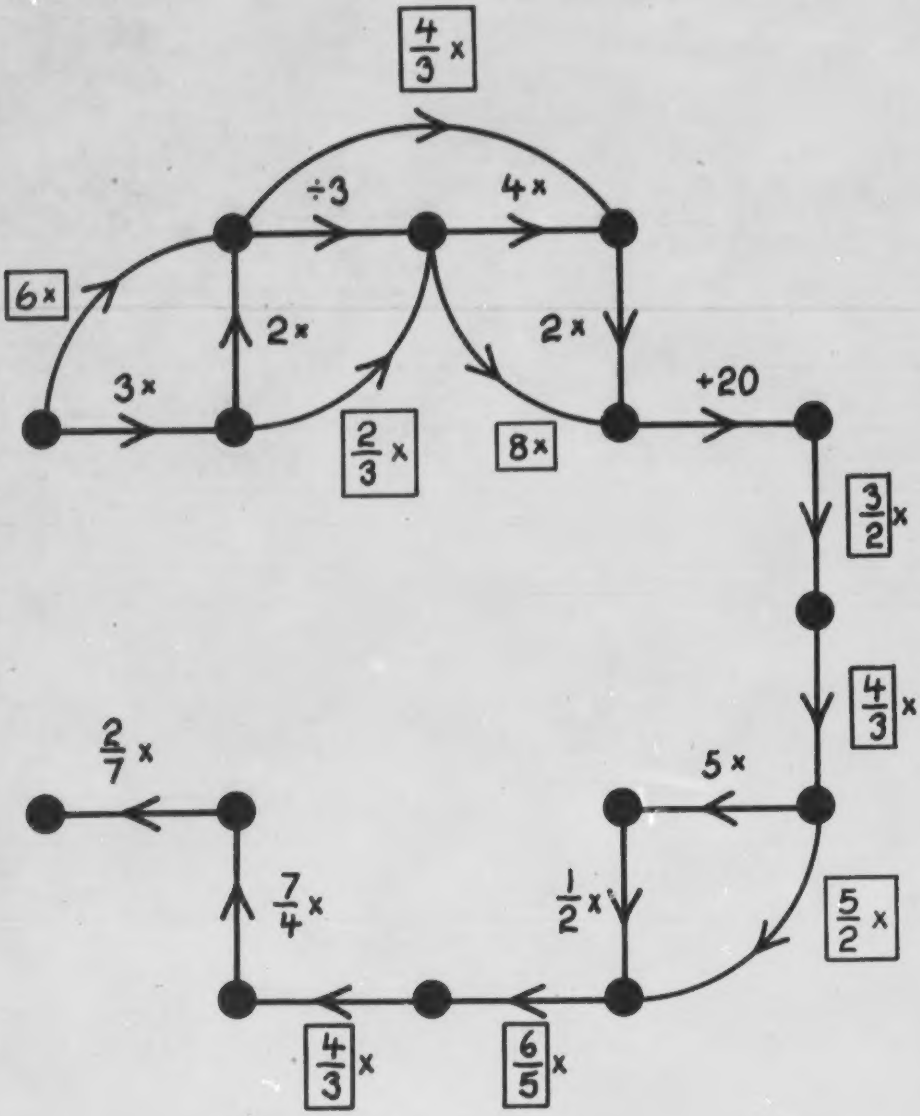
A completed arrow picture is drawn below with correct answers in the boxes.



30
÷3
4 x
40
10
2 x
2 x
3 x
+20
100
5
15
80
3/2 x
150
4/3 x
2/7 x
1,400
1,000
5 x
400
200
7/4 x
1/2 x
600
800
500
4/3 x
6/5 x

9N

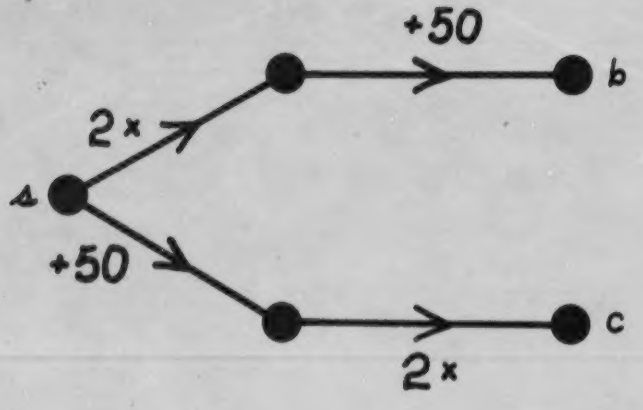
Erase the labels for the dots and any blue "detour" arrows in your arrow picture. Do not erase the red arrow labels. Draw these five blue arrows in your picture and have students label them. The answers are in the boxes; let the students write them on the board.



4/3 x
÷3
4x
6x
2x
2x
3x
+20
2/3 x
8x
3/2 x
4/3 x
2/7 x
5x
7/4 x
1/2 x
5/2 x
4/3 x
6/5 x

Exercise 2

Draw this arrow picture on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: All the dots in this picture are for positive numbers. (Point to b and c.) Copy this arrow picture on your paper and circle the dot that is for the larger number.

Let the students have a minute or two to work on this problem; then ask a student to point to the appropriate dot (the lower one).

T: Can someone explain why the lower dot is for the larger number?

S: The number is larger, because 50 is added first and then the sum is doubled.

Accept any reasonable explanation even if it is incomplete. The problem will be more completely discussed with further questioning.

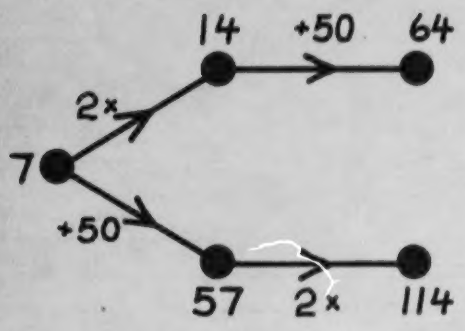
T: How much greater is the number at the bottom than the number at the top?

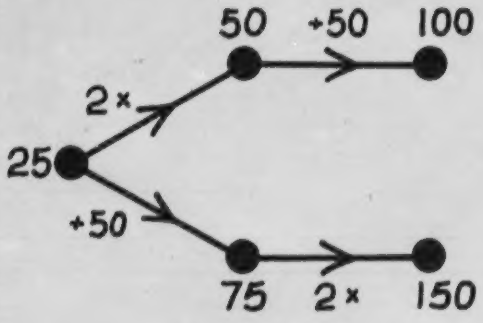
S: 50 greater.

T: Why?

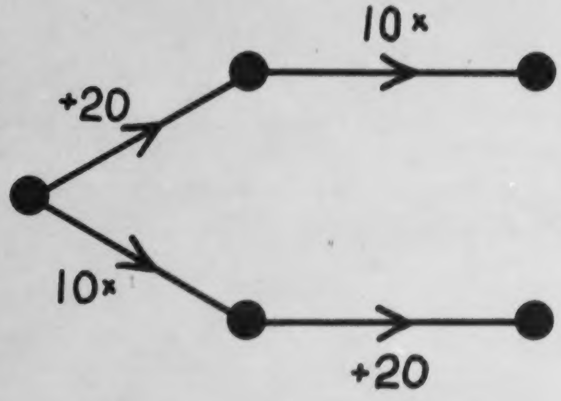
S: In the lower part of the picture, we first add 50 to a number and then double, giving the double of the number plus 100. In the upper part of the picture, we double the number and then add 50. So the lower number is 50 more than the upper one.

There will probably be a few students that are not convinced by such arguments. Present some examples for their benefit. Have a student label the dot at s and have other students complete the picture. This activity should be repeated two or three times. Two examples of a completed arrow picture are shown below.



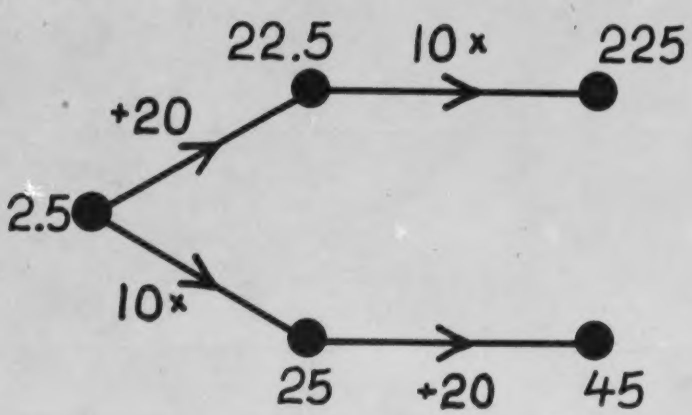


Draw this arrow picture on the board.



T: All the dots in this picture are for positive numbers, and 225 is the largest. Copy this picture on your paper and label the dots.

Give the students a few minutes to work on this problem independently. Then, ask a student to point to the dot for 225. Let the students work independently for a few minutes longer labeling the other dots. Then complete the picture collectively. A completed arrow picture is drawn below.

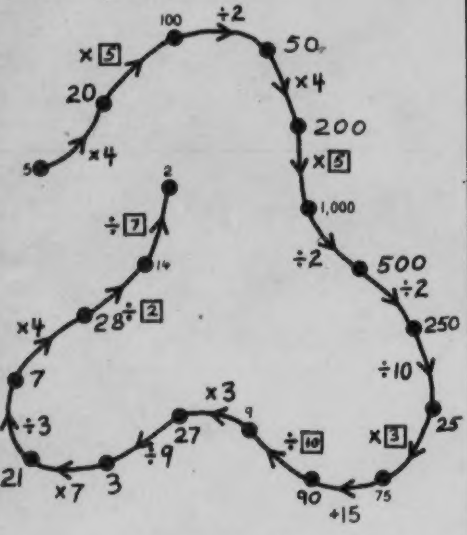


Worksheets N05, N06, N07, and N08 are available.

Name

N 05 *

Label the dots and fill in the boxes for the arrows.

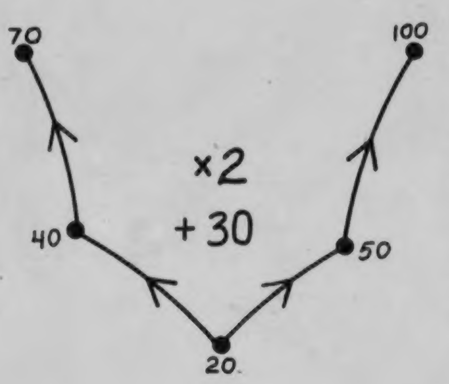


100
÷2
×5
50
×4
20
200
5
×4
×5
2
1,000
÷7
14
÷2
500
×4
28
÷2
÷2
250
7
÷10
×3
9
÷3
27
÷10
×3
25
21
×7
3
÷9
90
75
+15

Name

N 06 **

In this picture, all the dots are for POSITIVE numbers and 100 is the largest number.
Put 100 in the picture and label the dots.

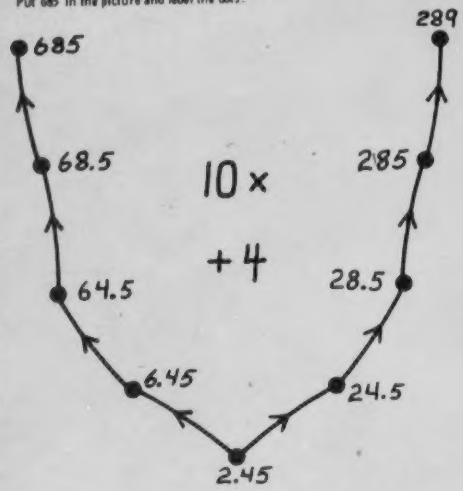


70
100
×2
+30
40
50
20

Name

N 07 ***

In this picture, all the dots are for POSITIVE numbers and 685 is the largest number.
Put 685 in the picture and label the dots.



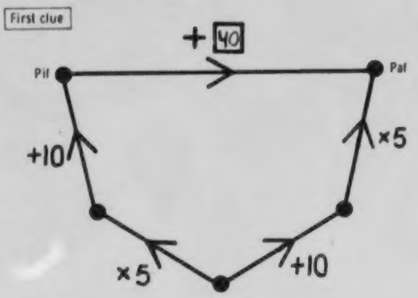
289
685
68.5
10×
285
+4
64.5
28.5
6.45
24.5
2.45

Name

N 08 ****

Pif and Paf are two secret numbers.

First clue



+40
Pif
Paf
+10
×5
×5
+10

Second clue

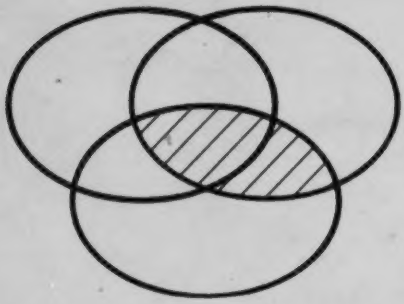
Paf is the double of Pif.

Pif is 40. Paf is 80.

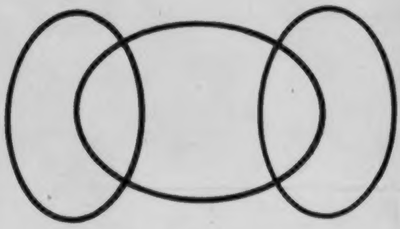
2

MULTIPLES AND DIVISORS

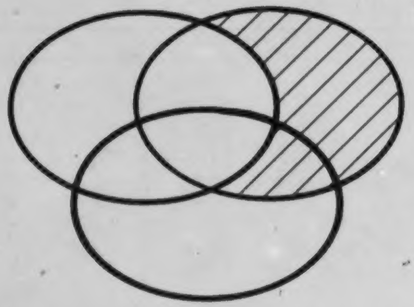
In this chapter, situations are presented in which the relation "is a divisor of" is reviewed using the language of strings. In the process, string pictures with hatching are found to have alternate forms; for example,



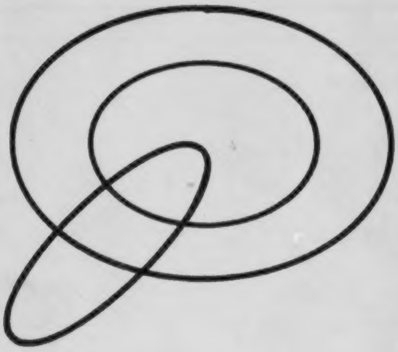
gives the same information as



and



gives the same information as



Using the relation "is a divisor of", certain numbers are identified as products of two distinct primes, others as squares of primes, etc.

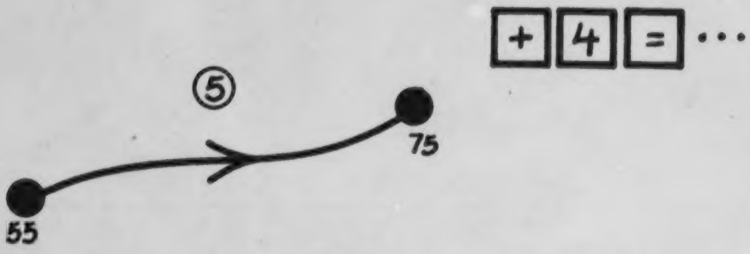
Multiples of numbers are encountered in situations in which hand-calculator relations, such as $\boxed{+}$ $\boxed{4}$ $\boxed{=}$ . . . , are employed. Before teaching the lesson on hand-calculator relations, you should read again Section 2(f) of the General Introduction on the Role and Use of Hand-calculators.

CAPSULE LESSON SUMMARIES

Section 2.1: Locate numbers in a string picture where the string labels are attributes such as: "positive divisors of □"; "square numbers"; "positive prime numbers"; "multiples of □"; "can be put on the Minicomputer using exactly one of these checkers: ② ... ⑨". Hatch the resulting string pictures where appropriate and determine alternative string pictures that require no hatching.
(Worksheets N09, N10, N11, N12)

Section 2.2: Among the positive numbers with exactly four positive divisors, determine which are products of two distinct primes and which are cubes of primes. Observe that all positive numbers with exactly three positive divisors are squares of primes.

Section 2.3: Review hand-calculator relations using $\boxed{+}$ $\boxed{4}$ $\boxed{=}$ ..., and introduce a notation for the number of times $\boxed{=}$ is pressed. For example,

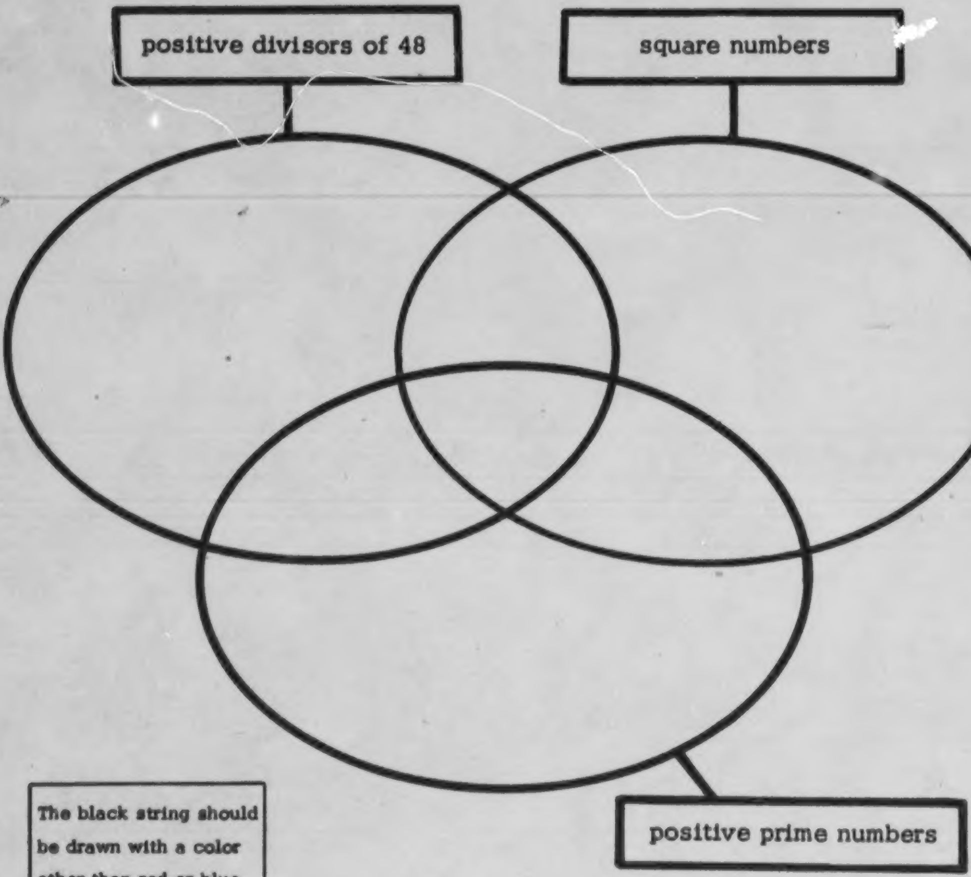


That is, put "55" on the display; then press $\boxed{+}$ $\boxed{4}$ and the $\boxed{=}$ fives times; the result is that "75" is on the display. Draw arrow roads with these "weighted" arrows between specified numbers, and determine the weights of composites.
(Worksheets N13, N14, N15, N16)

2.1 String Pictures

Exercise 1

Draw this string picture on the board.


positive divisors of 48
square numbers
positive prime numbers

The black string should be drawn with a color other than red or blue.

T: Can you give me some examples of positive divisors of 48?

S: 6, 24, 1.

Accept any positive divisor of 48; they are 1, 2, 3, 4, 6, 8, 12, 16, 24, and 48.

2N

T: Give me some examples of square numbers.

S: 9, because 3 × 3 = 9.

S: 25, because 5 × 5 = 25.

Accept any square number; e.g., 4, 9, 16, 25, 36, 49, 64, 81, 100, and so on.

T: Give me some examples of positive prime numbers?

S: 3, 7, 13.

T: Can you name some prime numbers larger than 20?

S: 23, 29.

T: Do you know some prime numbers larger than 30?

S: 37.

S: 53.

T: Let's put 25 in our string picture. Is 25 a positive divisor of 48?

S: No.

T: Is 25 a square number?

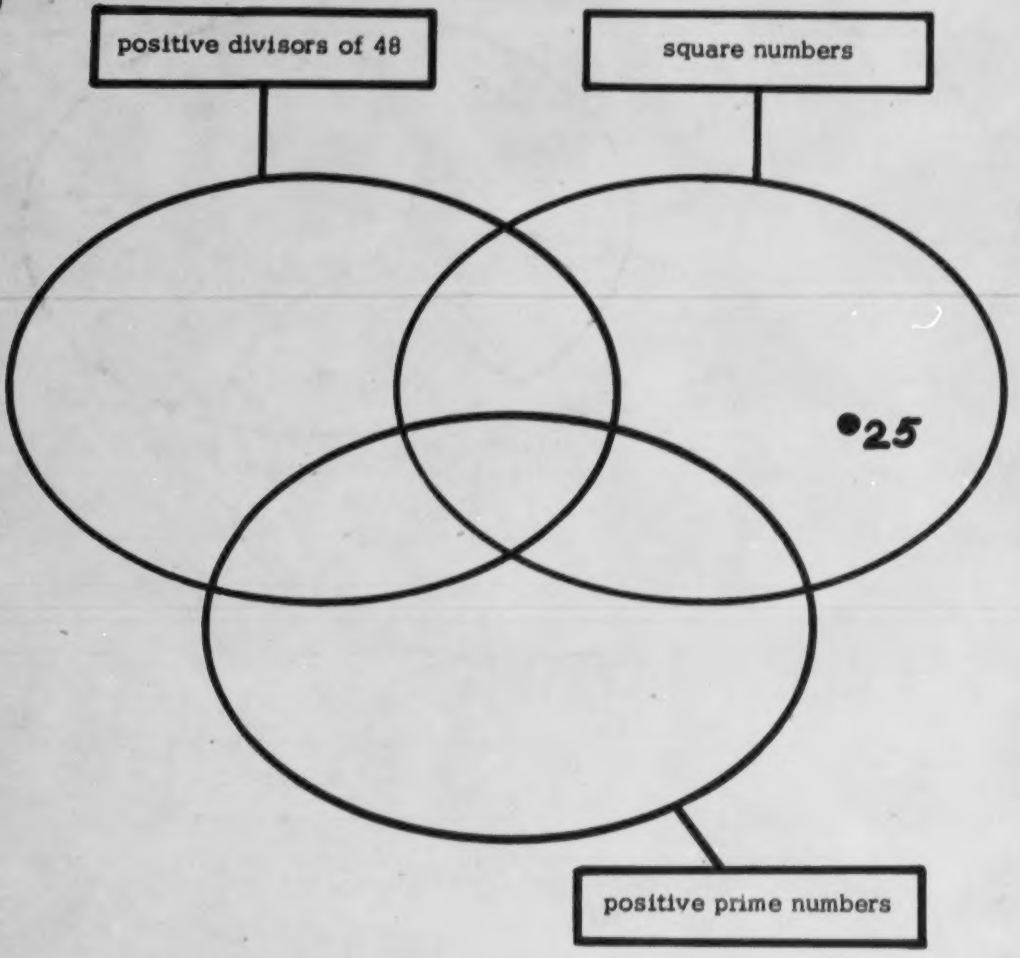
S: Yes, 5 × 5 = 25.

T: Is 25 a positive prime number?

S: No.

T: Where is 25 in the string picture?

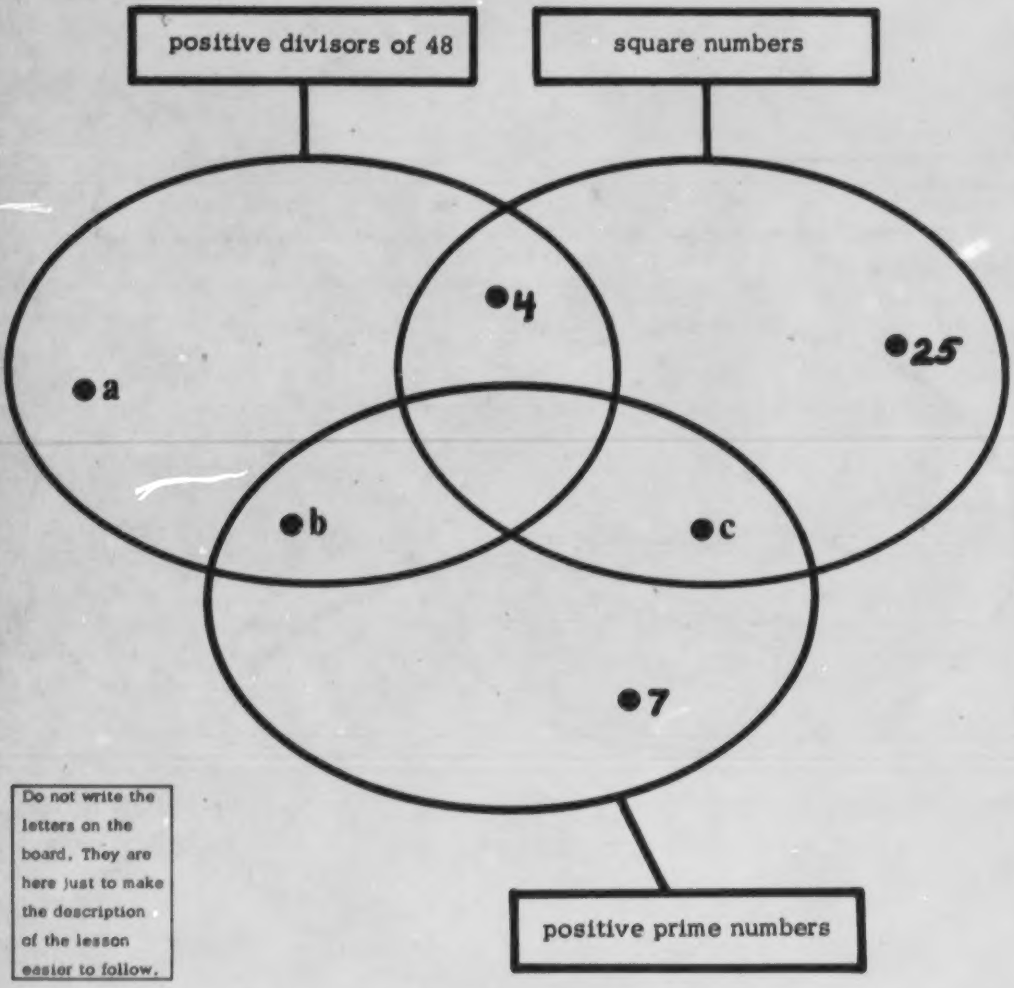
Ask a student to locate and label a dot for 25.


positive divisors of 48
square numbers
25
positive prime numbers

Continue in this manner asking students to put 4 and 7 in the string picture.

Add the dot labeled "a" (see the next illustration) to your picture. Dots labeled "b" and "c" will be added later in the lesson.

2N


positive divisors of 48
square numbers
4
a
25
b
c
7
positive prime numbers

Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T (pointing to a): What numbers could this dot be for?

Accept as correct any positive divisor of 48 that is neither a square number nor a positive prime number; e.g., 6, 8, 12, 24, or 48.

Add the dot labeled "b" to your picture and continue as above. Accept as correct any positive divisor of 48 that is a positive prime number and is not a square number; e.g., 2 or 3.

Add the dot labeled "c" to your picture.

T (pointing to c): What could this dot be for?

It is quite likely that your students' first responses to this question will be incorrect. For example,

S: 9.

T: Is 9 a square number?

S: Yes.

T: Is 9 a prime number?

S: No.

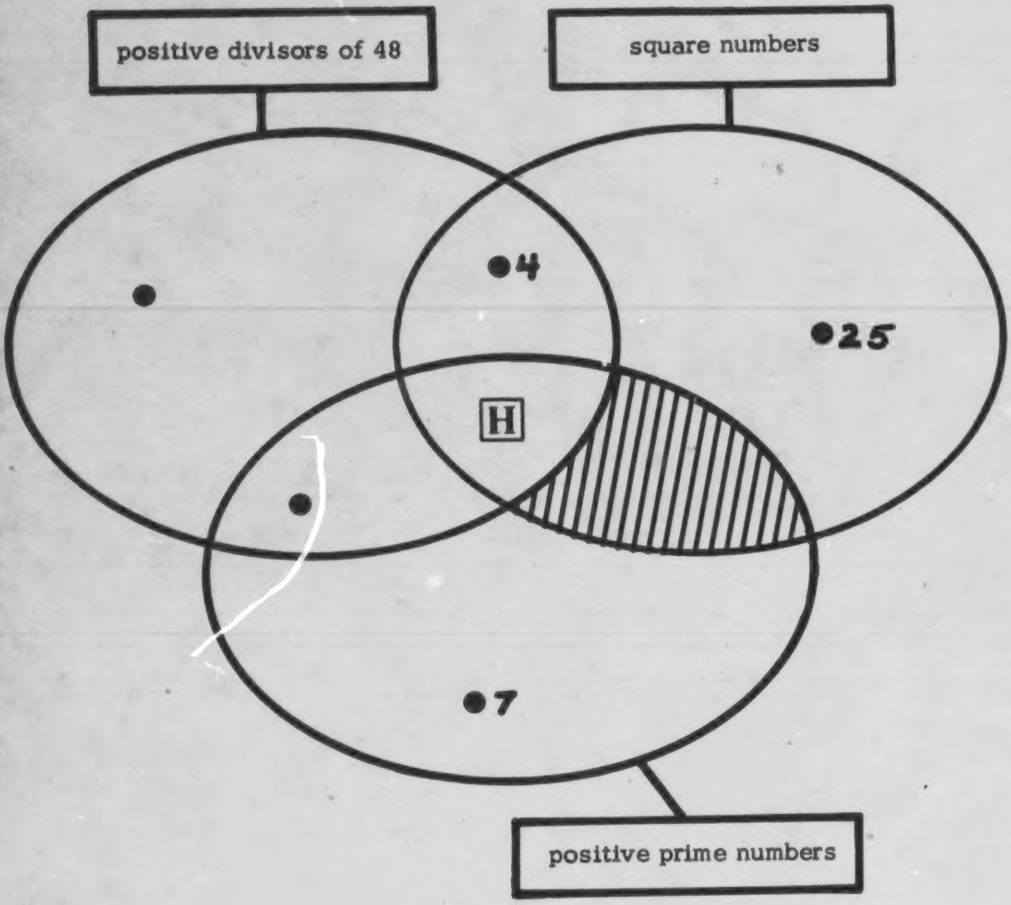
T: Then, could this dot be for 9?

S: No.

After several examples someone will surely suggest that there is nothing the dot could be, because square numbers are not prime. The region should be hatched.

2N

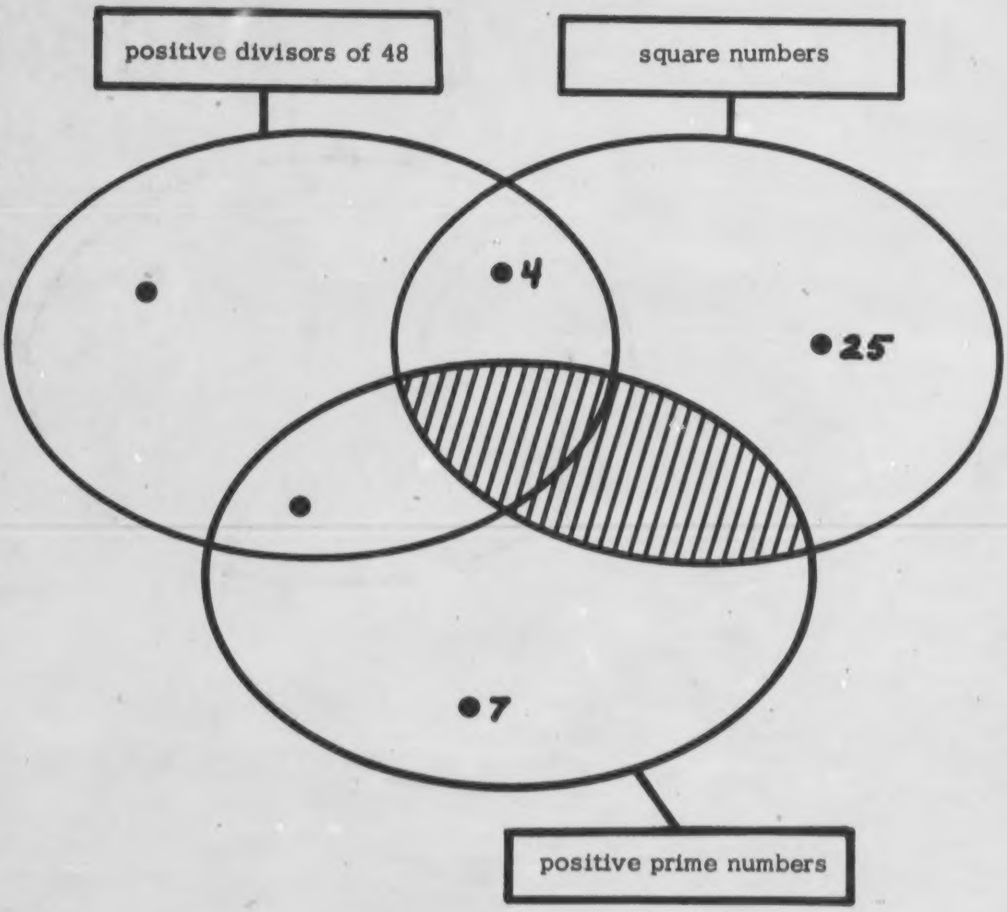
Invite a student to hatch the empty region. More than likely the student will hatch the region shown below. The empty middle region, H in the drawing below, is easily overlooked by some students.


positive divisors of 48
square numbers
4
25
H
7
positive prime numbers

T: You said that no square number is prime. Isn't there another region that should be hatched?

After a few false starts, someone should suggest that region H be hatched.

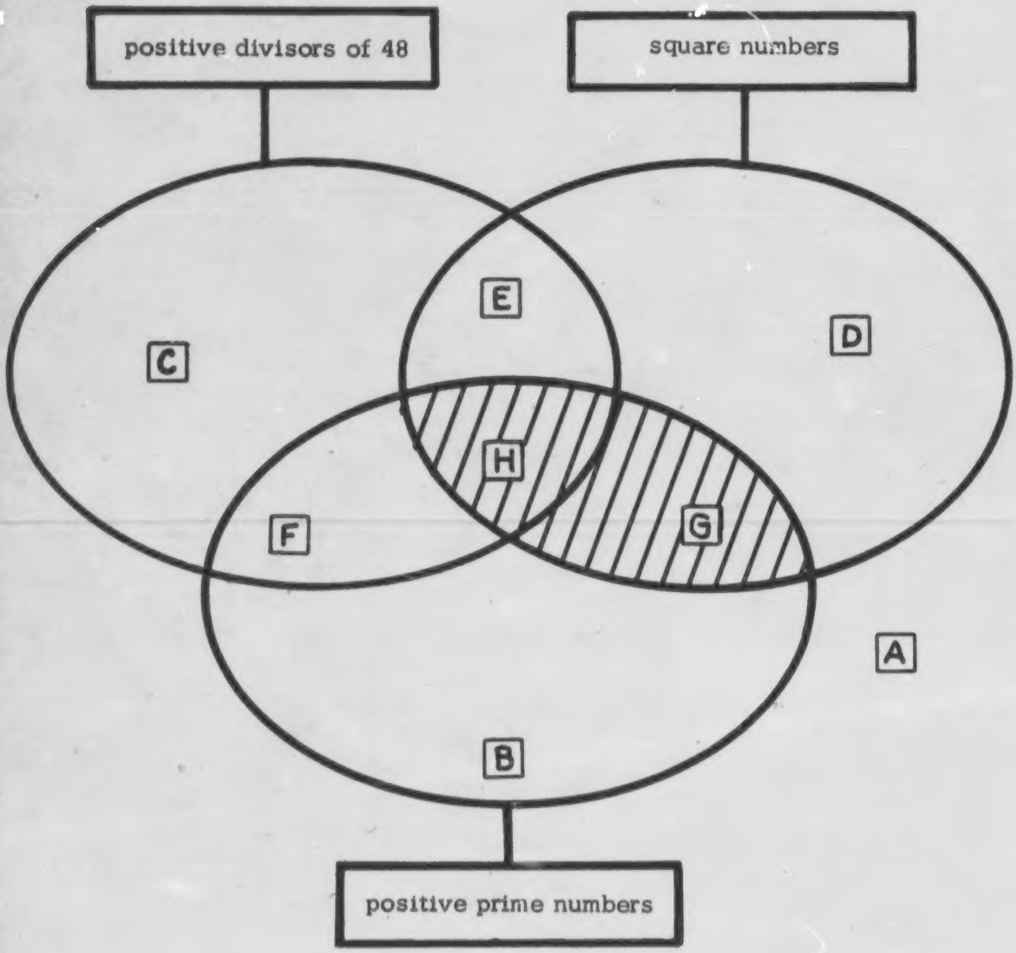
Your string picture should look like this.


positive divisors of 48
square numbers
4
25
7
positive prime numbers

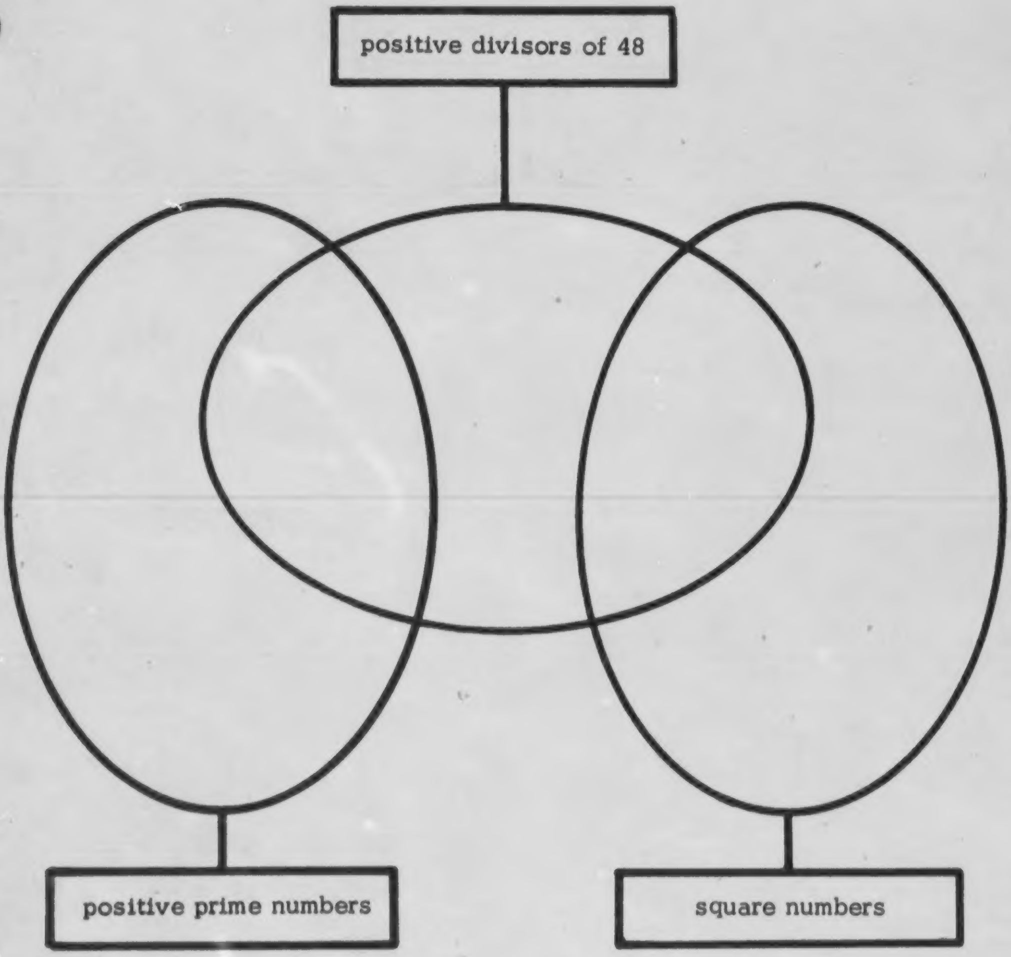
T: Can we draw this picture another way without using any hatching?

2 N

Let the students work independently on this situation. While the students are working, label the regions of your picture as shown below.


positive divisors of 48
square numbers
C
E
D
H
F
G
A
B
positive prime numbers

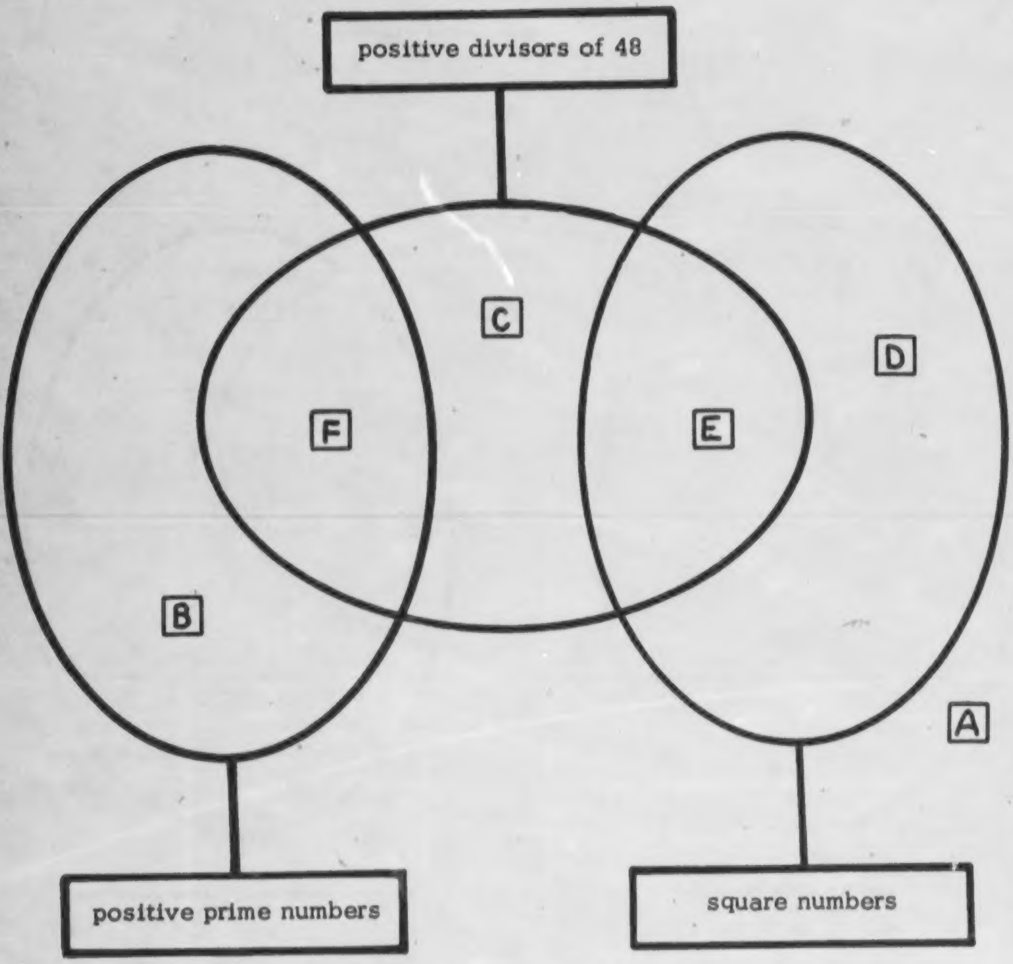
Ask a student who has found another way, to draw the picture on the board. Someone should suggest a picture similar to this one.


positive divisors of 48
positive prime numbers
square numbers

T: Look at region A of the picture I drew. Who can label region A in the new picture?

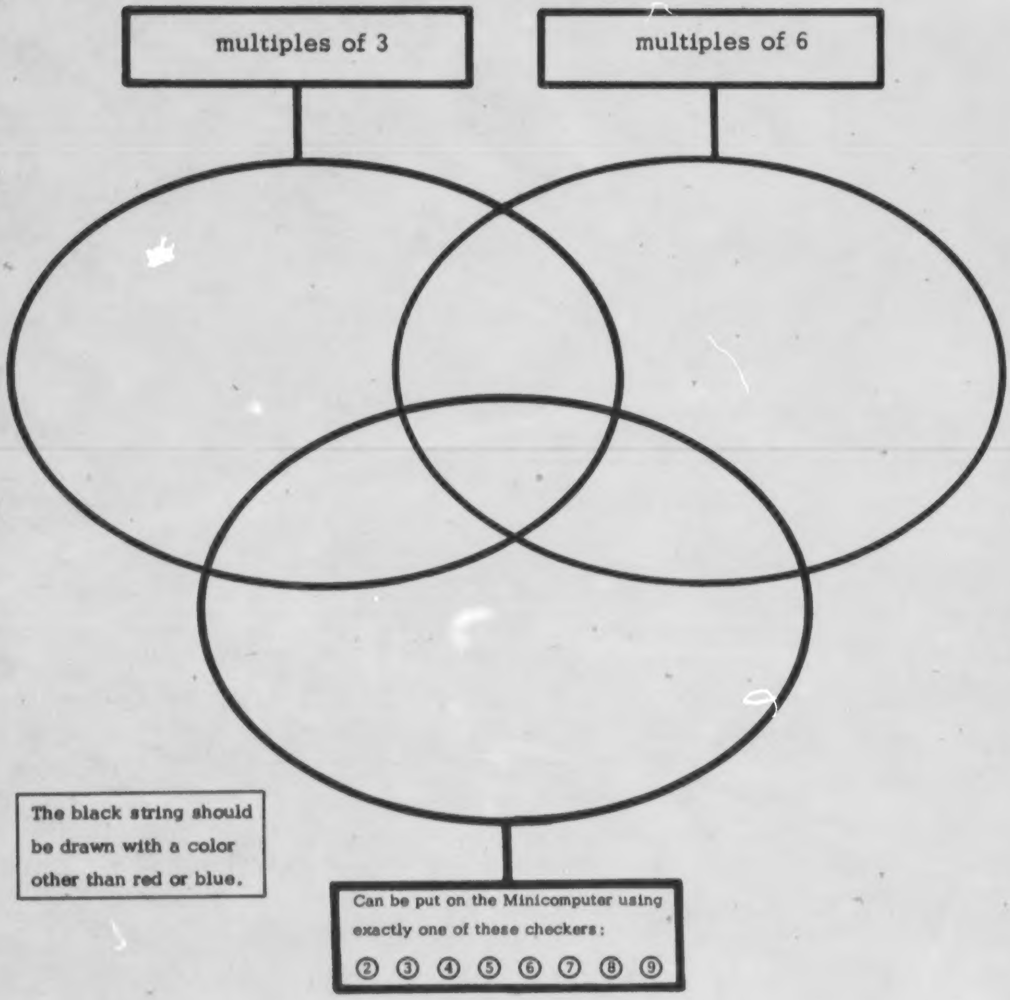
2N

Have students label the regions of the new picture. A completed string picture is drawn below.


positive divisors of 48
C
D
F
E
B
A
positive prime numbers
square numbers

Exercise 2

Draw this string picture on the board.


multiples of 3
multiples of 6
Can be put on the Minicomputer using exactly one of these checkers:
② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

The black string should be drawn with a color other than red or blue.

T: On your paper, list at least six multiples of 3. Try to find at least one multiple of 3 greater than 100.

Check some of the students' responses collectively. Accept any multiple of 3; e.g., $^{-}9$; $^{-}3$; 0; 6; 18; 66; 69; 102; 1,020; and so on.

2N

T: On your paper, list at least six multiples of 6. Try to find at least one multiple of 6 greater than 100.

Again, check some responses collectively. Accept any multiple of 6; e.g., ¯18; ¯12; ¯6; 0; 6; 12; 48; 96; 102; 1,020; and so on.

Display two Minicomputer boards and one each of the new checkers:

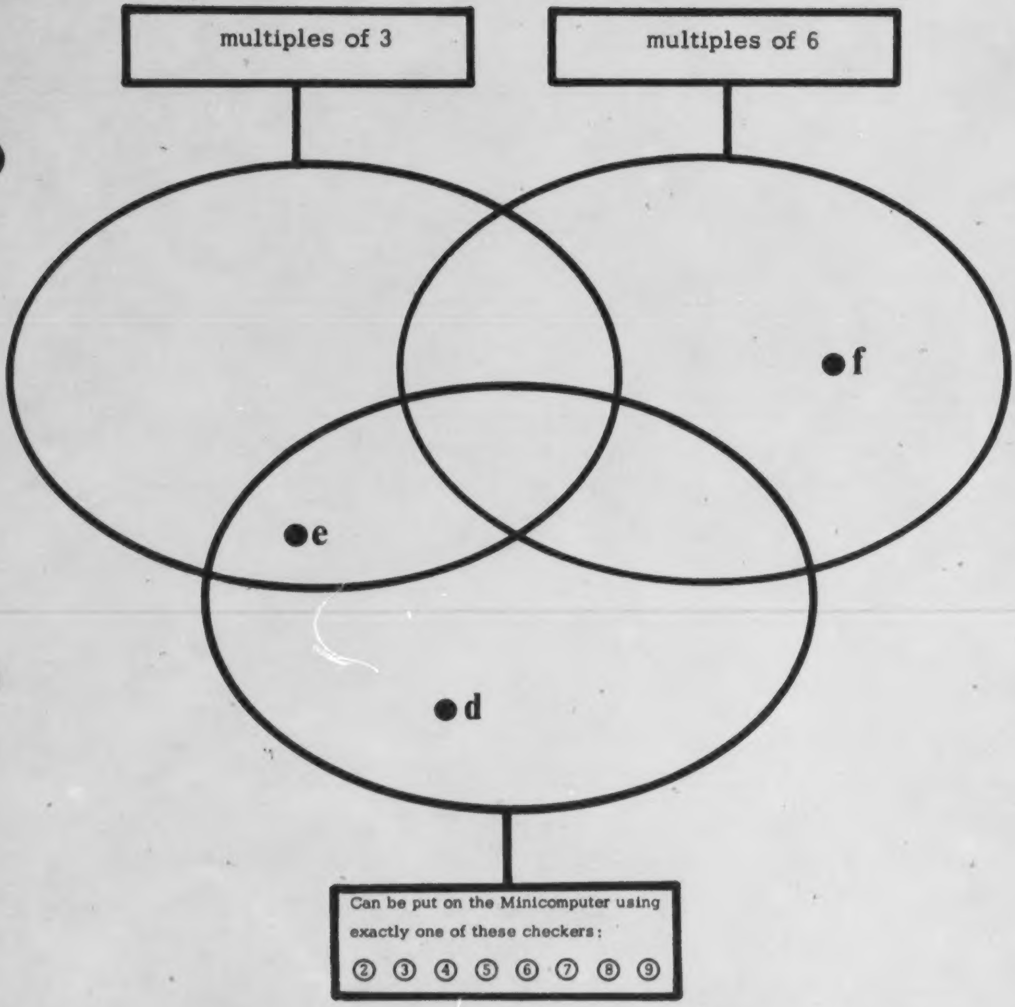
② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

T: On your paper, list at least four numbers that can be put on the Minicomputer using exactly one of the new checkers, ② through ⑨. I have displayed only two Minicomputer boards, but you may use more than two if you wish.

Invite several students to put some numbers on the Minicomputer with exactly one new checker. Some possible responses are 36; 500; 40; 140; 10; and so on.

Add the dot labeled "d" (see the next illustration) to your picture. Dots labeled "e" and "f" will be added later.

2N


multiples of 3
multiples of 6
f
e
d
Can be put on the Minicomputer using exactly one of these checkers:
② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

T (pointing to d): What could this dot be for? Is this number a multiple of 3?

S: No, it is outside the red string.

T: Is this number a multiple of 6?

S: No, it is outside the blue string.

T: On your paper, list some numbers that could be here.

S: 16.

Have a student put 16 on the Minicomputer using exactly one of the new checkers. Be sure the class checks that 16 is neither a multiple of 3 nor a multiple of 6. Continue in this manner for other numbers that are suggested. Some numbers that d could be are 32, 320, 5, 2, 4, and so on.

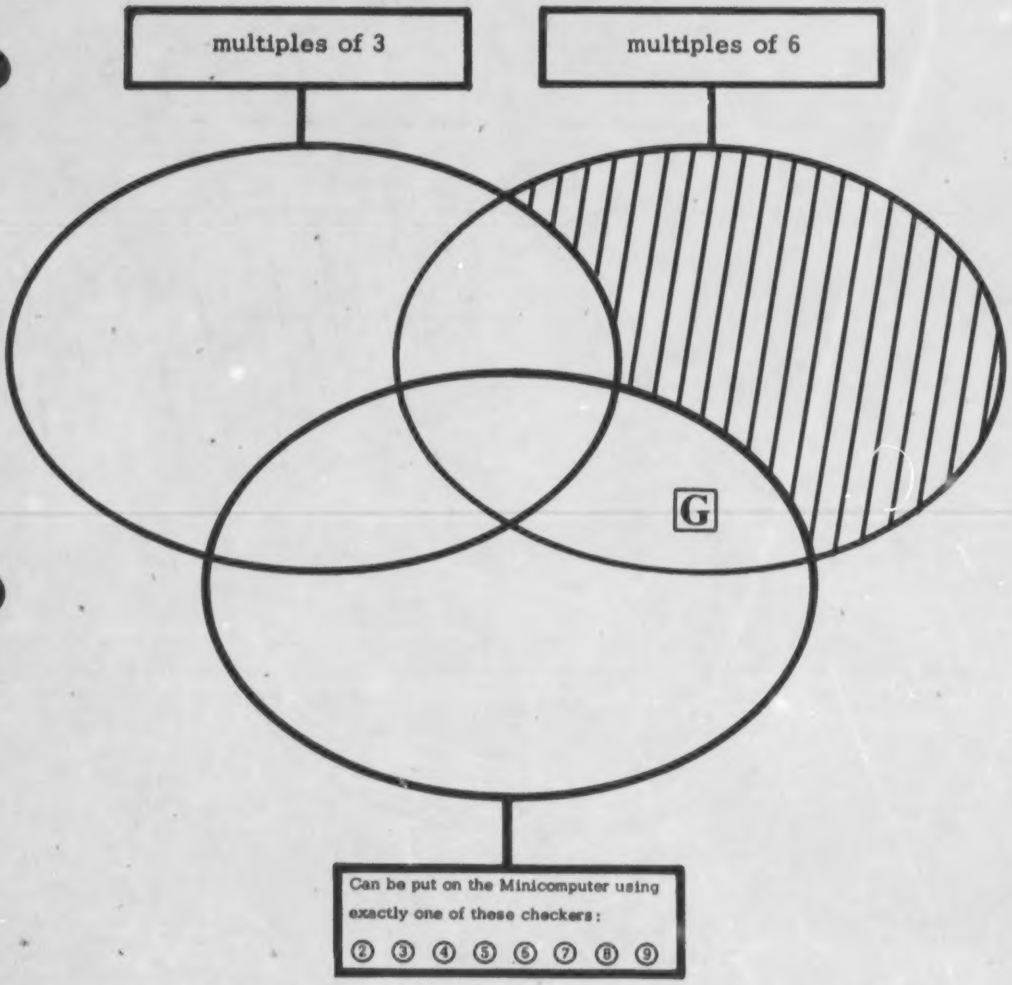
Add the dot labeled "e" to your picture and continue as above. There are exactly two numbers that belong in this region — 3 and 9.

Add the dot labeled "f" to your picture and ask:

T: What could this dot be for?

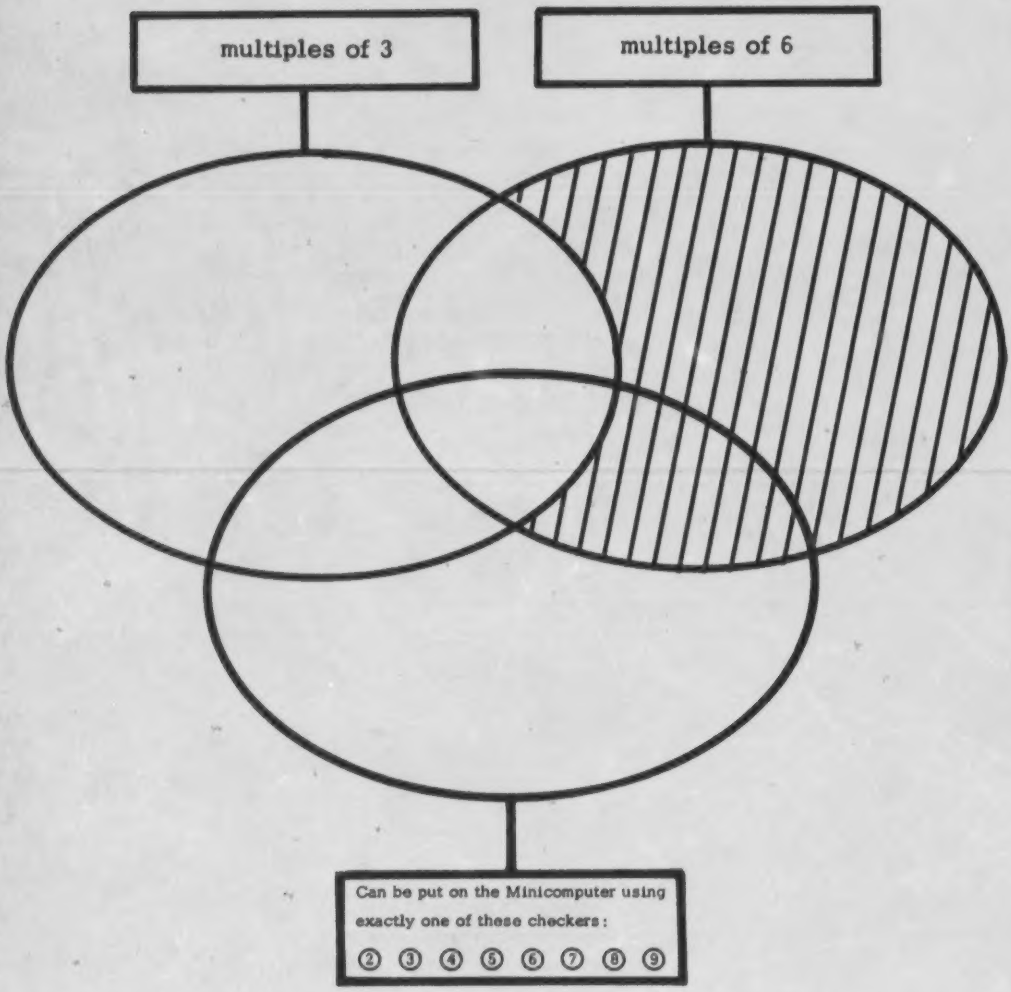
As in Exercise 1, you are quite likely to receive a few incorrect responses before students recognize that the region is empty and should be hatched. All multiples of 6 are multiples of 3 and should be included within the red string.

A student might hatch the region as shown below, omitting region G.


multiples of 3
multiples of 6
G
Can be put on the Minicomputer using exactly one of these checkers:
② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

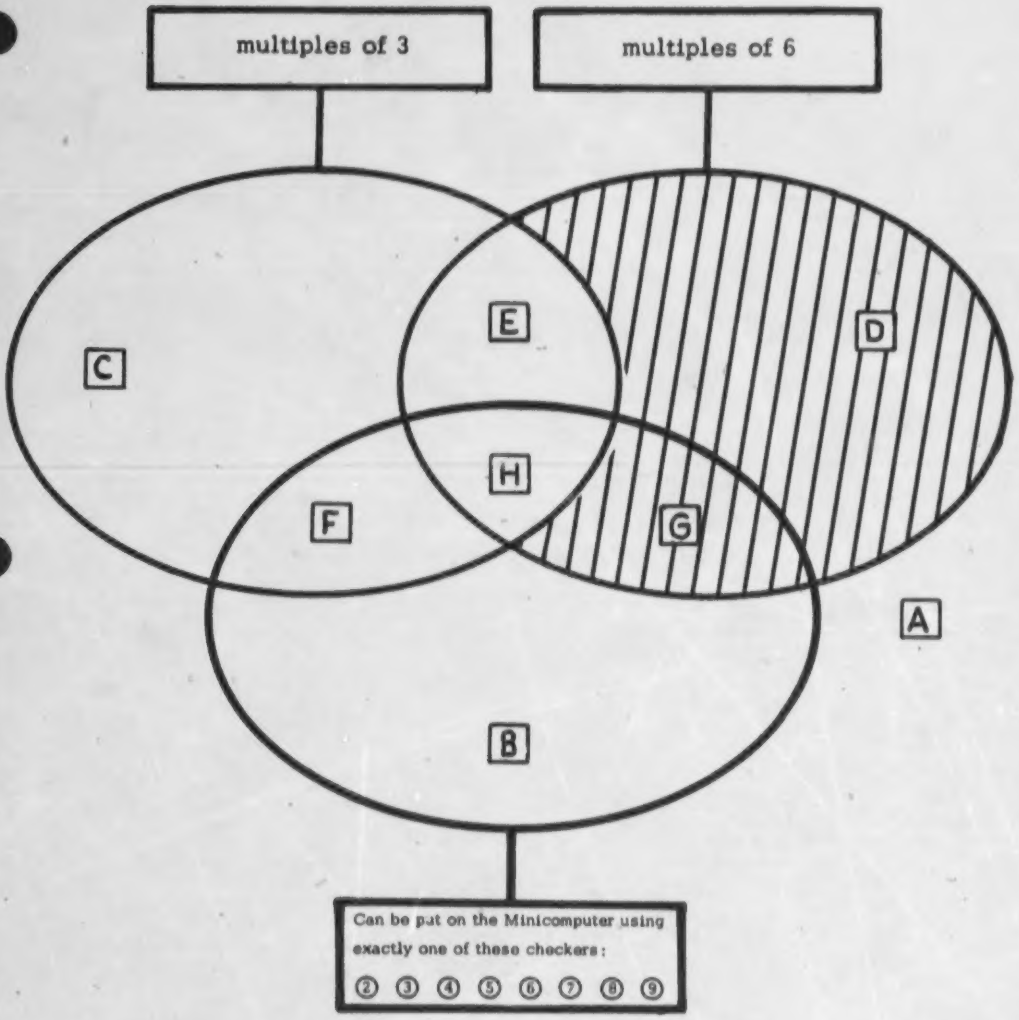
2 N

As in Exercise 1, ask about the hatching of region G, and then complete the hatching as below.


multiples of 3
multiples of 6
Can be put on the Minicomputer using exactly one of these checkers:
② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

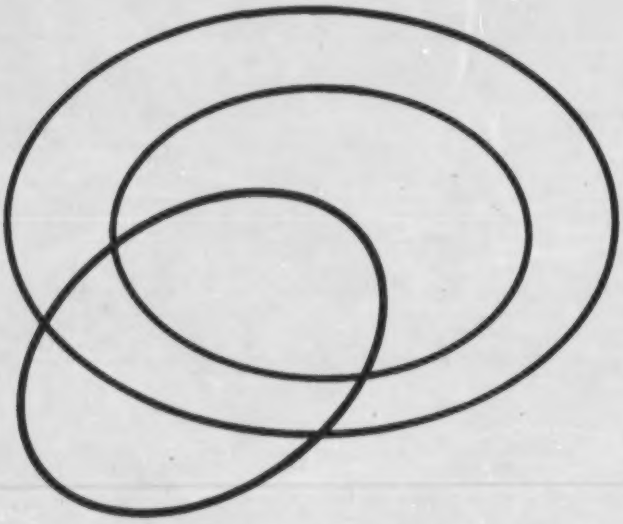
T: Can we draw this picture another way without using any hatching?

Let the students work independently on this situation. While the students are working, label the regions of your picture as shown below.


multiples of 3
multiples of 6
C
E
D
H
F
G
A
B
Can be put on the Minicomputer using exactly one of these checkers:
② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

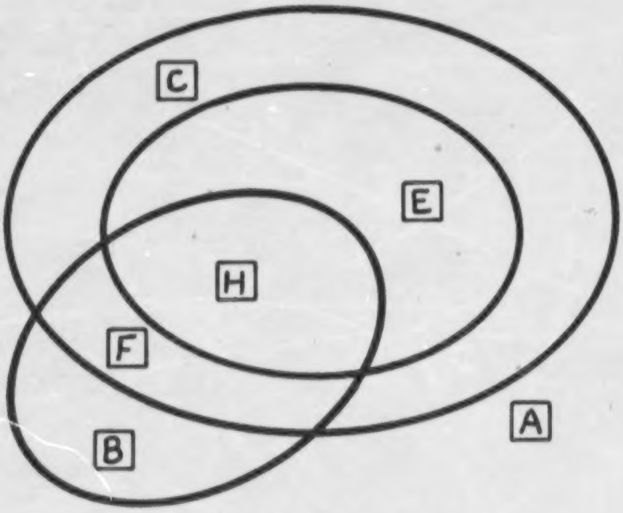
2N

Ask a student who has found another way to draw the picture, to put it on the board. It should look similar to this one.



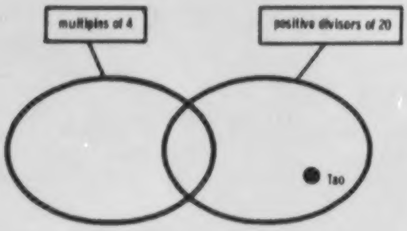
T: Look at region A of the picture I drew. Who can label region A in the new picture?

Ask students to label the regions of the new picture. A completed string picture is drawn below.



Worksheets N09, N10, N11, and N12 are available.

Name ____________ **N 09** *

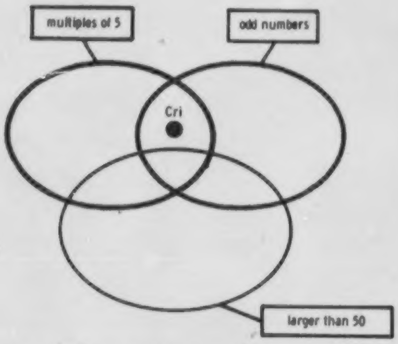

multiples of 4
positive divisors of 20
Tao

Can Tao be 10? YES; 4? NO; 8? NO; 20? NO;

6? NO; 5? YES; 2? YES; 7? NO

Tao is larger than 5. Who is Tao? 10

Name ____________ **N 10** **

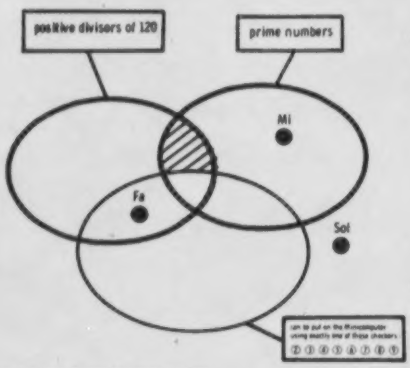

multiples of 5
odd numbers
Cri
larger than 50

Can Cri be 75? NO; 20? NO; 15? YES; 50? NO;

8? NO; 35? YES; 105? NO; 72? NO

Cri is larger than 37. Who is Cri? 45

Name ____________ **N 11** ***


positive divisors of 120
prime numbers
Mi
Fa
Sol
can be put on the Minicomputer using exactly one of these checkers:
② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

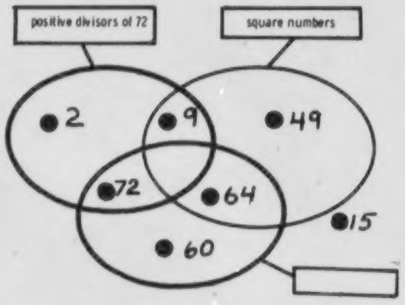
Mi, Fa and Sol are three of these numbers:

3; 5; 7; 15; 35; 36; 37; 60

Mi is 37; Fa is 60; Sol is 35.

One region of this diagram should be hatched. Hatch it.

Name ____________ **N 12** ****


positive divisors of 72
square numbers
2
9
49
72
64
15
60

The blue tag is one of these:

larger than 50

multiples of 4

With this information you should be able to place seven of these numbers in the string picture.

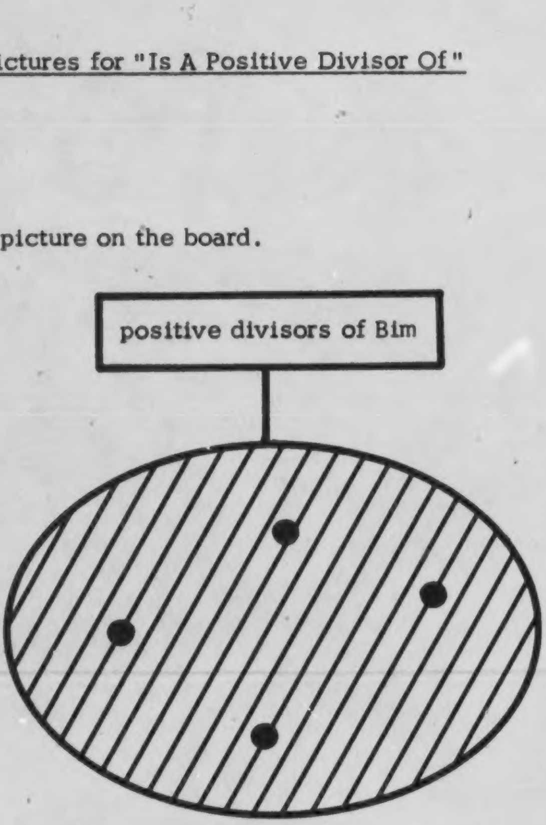
2; 4; 9; 15; 20; 49; 60; 64; 72.

Place them.

2.2 Arrow Pictures for "Is A Positive Divisor Of"

Exercise 1

Draw this string picture on the board.



T:	Bim is a secret whole number. What does the string picture tell us about Bim?
S:	Bim has exactly four positive divisors.
T:	What is one number that Bim could be?
S:	6.
T:	What are the positive divisors of 6?
S:	1, 2, 3, and 6.
T:	6 has exactly four positive divisors, so Bim could be 6. On your paper, list some other numbers that Bim could be.

Let the students have a few minutes to work independently. Encourage them to find several numbers that Bim could be.

After a few minutes, bring the class together for a collective discussion of numbers that Bim could be. For each number that is suggested, ask a student to verify it by giving the positive divisors of the number. It would be helpful to make a list on the board of the numbers that Bim could be — especially for the latter part of this activity.

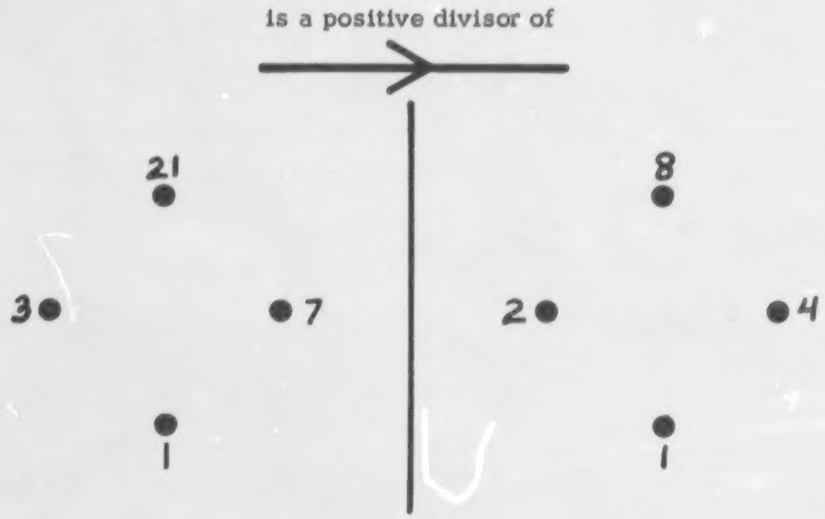
Suppose students have suggested the following numbers: 6, 8, 14, 10, 15, 21, 27, 35, 22, and 33.

NOTE: It is crucial to the following discussion that both 8 and 27 be among the numbers suggested for Bim. If you find that 27 has been overlooked, ask:

T: Have we found all of the numbers between 20 and 30 that Bim could be?

S: Bim could be 21, 22, 26, and 27.

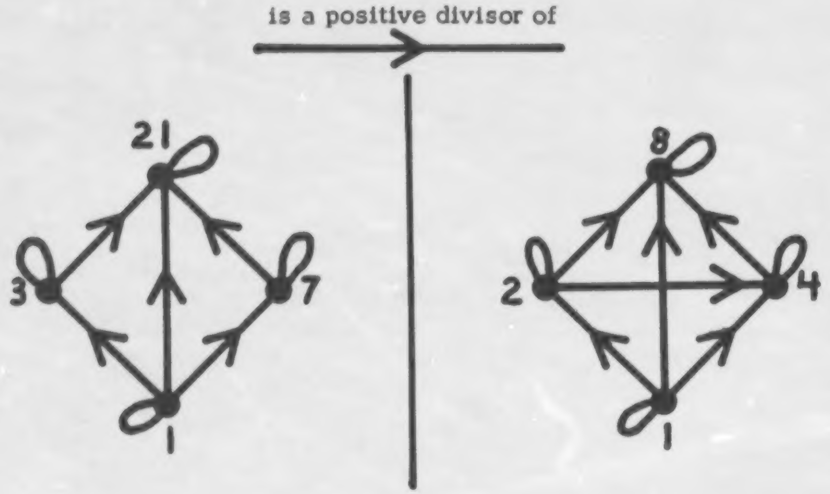
T: On the left side of the board, I will draw the four positive divisors of 21. On the right side of the board, I will draw the four positive divisors of 8.



23N

T: Complete each arrow picture. The red arrow is for "is a positive divisor of".

While the students are working independently, ask some students to complete the pictures at the board. Completed pictures are drawn below.



T: Compare the two pictures.

S: They are different.

T: How are they different?

S: In the arrow picture for 8, 2 is a positive divisor of 4. In the arrow picture for 21, 3 is not a positive divisor of 7.

Accept any reasonable explanation pointing out that in the picture on the right, there is an arrow between each pair of divisors. This is not the case in the picture on the left.

T: Now choose any two other numbers from our list and make an arrow picture for them. Try to find one with an arrow picture like the one on the left and another with an arrow picture like the one on the right.

Let the students have several minutes to work independently. Encourage the faster workers to draw several more arrow pictures. After a while, bring the class back together for a collective discussion of this situation.

T: We will circle in green each number that has an arrow picture like 21 and circle in blue each number that has an arrow picture like 8.

The class should classify the numbers that Bim could be as green or blue. If necessary, some of the arrow pictures could be drawn on the board. Be sure to circle each number with its appropriate color.

In our list of numbers that Bim could be, 8 and 27 are blue numbers and the others are green numbers.

T: Do you see any difference between the blue numbers and the green numbers?

Accept any reasonable comment; however, you may want to be more explicit about what differences the students should look for. For example, say:

T: I will give you some hints.

Write this number sentence on the board.

$$14 = \square \times \triangle$$

T: Can you write 14 as the product of two prime numbers?

S: 2 × 7.

23N

Write the remaining green numbers on the board and ask the students to express each number as the product of two prime numbers. Be sure to record the products on the board. For example,

$14 = 2 \times 7$

$6 = 2 \times 3$

$10 = 2 \times 5$

$15 = 3 \times 5$

$21 = 3 \times 7$

$35 = 5 \times 7$

$22 = 2 \times 11$

$33 = 3 \times 11$

T: What can we say about the green numbers?

S: Each green number is the product of two different prime numbers.

T: What about the blue numbers? Can 8 be written as the product of two prime numbers?

S: No; 8 = 2 x 4, but 4 is not a prime number.

T: Can 8 be written as the product of some prime numbers? The prime numbers do not have to be different.

S: Yes; 8 = 2 × 2 × 2.

T: Can 27 be written as the product of some primes?

S: Yes; 27 = 3 × 3 × 3.

T: What can we say about the blue numbers?

S: Each blue number is the product of a prime number times itself and then again times itself.

In fact, each blue number is the cube or third power of a prime number. Accept any explanation along this line.

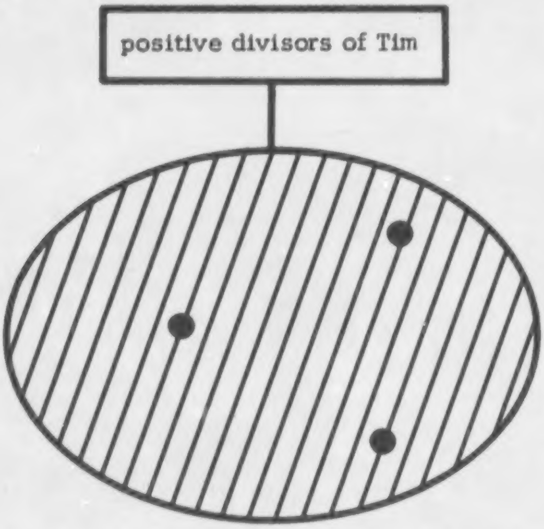
T: Can you find at least one more blue number and at least one more green number?

Call upon a few students to tell you the blue numbers and the green numbers they have found. 125 is a likely blue number that will be found.

Exercise 2

This exercise develops in the same spirit as Exercise 1.

Draw this picture on the board.



T: Tim is a secret whole number. What does the string picture tell us about Tim?

S: Tim has exactly three positive divisors.

T: What is one number that Tim could be?

73

S: 4, because the positive divisors of 4 are 1, 2, and 4.

T: On your paper, list some other numbers that Tim could be.

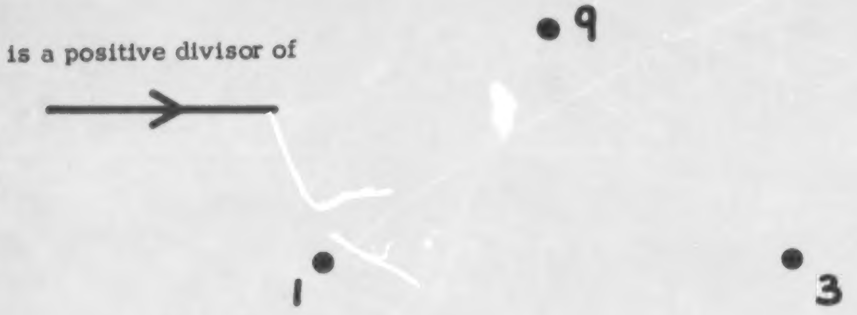
Encourage the students to find several numbers that Tim could be. As these numbers are found, list them on the board. It is likely students will find some of the following numbers: 4, 16, 25, 9, and 49.

Pick one of these numbers, for example 9.

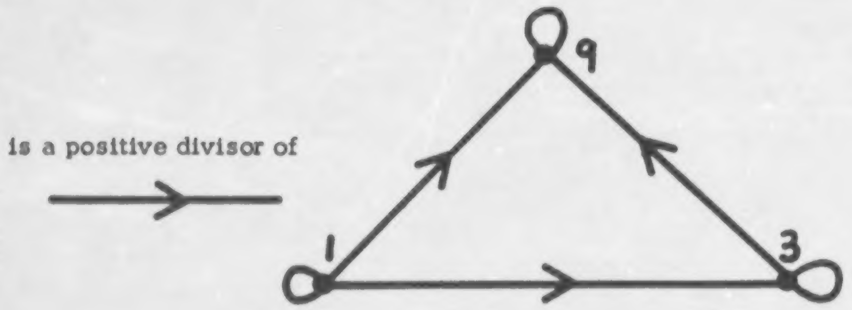
T: What are the positive divisors of 9?

S: 1, 3, and 9.

Draw this picture on the board.



Let the students complete the arrow picture on their paper and ask someone to complete the picture at the board. A completed picture is shown below.



Ask the students to draw arrow pictures for some of the other numbers that Tim could be.

S: All of the arrow pictures are the same.

T: Let's try to find a pattern for these numbers. Can you express 4 as a product of prime numbers?

S: $4 = 2 \times 2$.

T: How about 9?

S: $9 = 3 \times 3$.

Write the calculations on the board for emphasis. Continue this for each number that Tim could be. For example,

$$\begin{aligned} 4 &= 2 \times 2 \\ 9 &= 3 \times 3 \\ 25 &= 5 \times 5 \\ 49 &= 7 \times 7 \end{aligned}$$

T: What do you notice?

S: Each number is a square number.

T: Do you think Tim could be any square number?

S: No; 36 is a square number, but 36 has more than 3 divisors.

S: Tim is the square of a prime number.

23N

T: Very good. Can you give me some other numbers that Tim could be?

S: 121 = 11 x 11.

This activity can be continued as long as there is student interest.

2.3 Hand-Calculator Relations

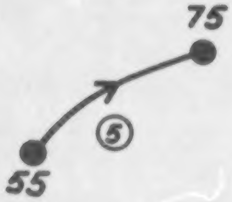
Exercise 1

Distribute as many hand-calculators as you have available. If there are only a few calculators, the students with calculators should answer any of the questions related to the calculator when the display is exposed. Any student can answer when the display is hidden. All students should be encouraged to think carefully and stay just as involved as if they had a calculator with a hidden display. The students with calculators can check all the answers. For different parts of the lesson, choose different students to work the calculators in order to give as many of them the experience as possible.

T: Display 55 on your hand-calculator. Hide the display and press $\boxed{+}\ \boxed{4}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}$ (press $\boxed{=}$ exactly 5 times). Which number is on the display? Why?

S: 75, because 5 x 4 = 20 and 55 + 20 = 75.

Draw this arrow picture on the board.



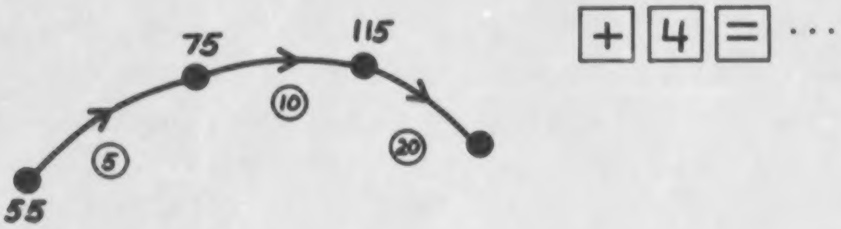
$\boxed{+}\ \boxed{4}\ \boxed{=}\ \cdots$

T: The arrow picture shows what we just did. The "⑤" tells us how many times we pressed $\boxed{=}$. Now press $\boxed{=}$ 10 more times. Which number is on the display? Why?

26N

S: 115, because 10 × 4 = 40 and 75 + 40 = 115.

Extend your arrow picture.



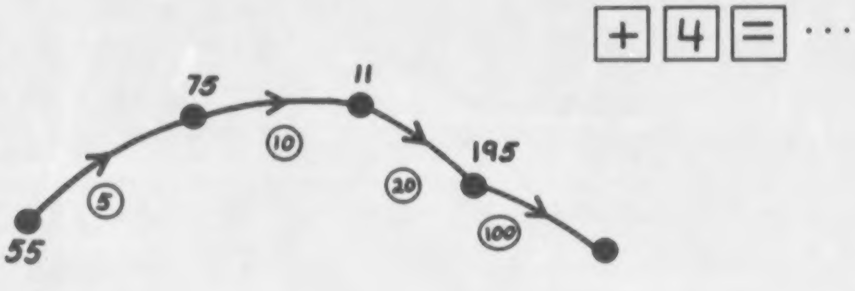
T: What does the arrow picture tell us to do?

S: Start with 115; press $\boxed{+}$ $\boxed{4}$ and $\boxed{=}$ 20 times.

T: Very good. Which number should be on the display? Why?

S: 195, because 20 × 4 = 80 and 115 + 80 = 195.

Add this arrow starting at 195 to your drawing.

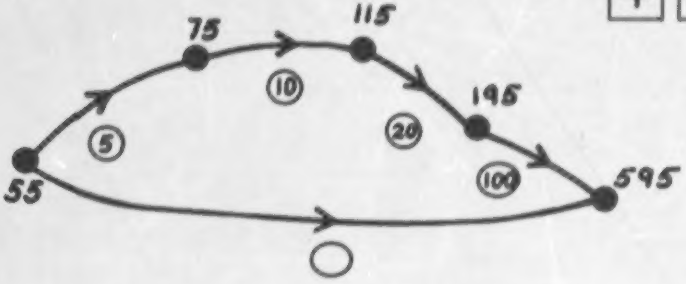


T: Don't press $\boxed{=}$; just write on your paper which number should be on the display.

S: 595, because 100 × 4 = 400 and 195 + 400 = 595.

Label the dot "595" and draw this arrow from 55 to 595.

$\boxed{+}\boxed{4}\boxed{=}\cdots$


75
115
195
55
595
⑤
⑩
⑳
(100)

T: What should we write inside the "○"?

S: 135.

T: Let's check. Calculate "135 × 4".

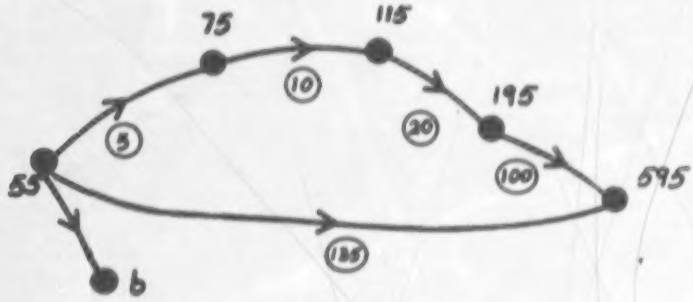
S: 540.

T: 55 + 540?

S: 595.

Add this arrow from 55 to b to your picture.

$\boxed{+}\boxed{4}\boxed{=}\cdots$


75
115
195
55
595
⑤
⑩
⑳
(100)
(135)
b

Do not write the letter on the board. It is here just to make the description of the lesson easier to follow

26 N

T (pointing to b): This is the largest possible number smaller than 1,000 I can get if I start from 55 and press $\boxed{+}$ $\boxed{4}$ $\boxed{=}$ . . . . What is it? Write the answer on your paper.

Let the students have a minute or two to consider this situation.

S: 999.

T: How did you find 999?

S: I know I can get 599 and 599 + 400 = 999. So I can get 999.

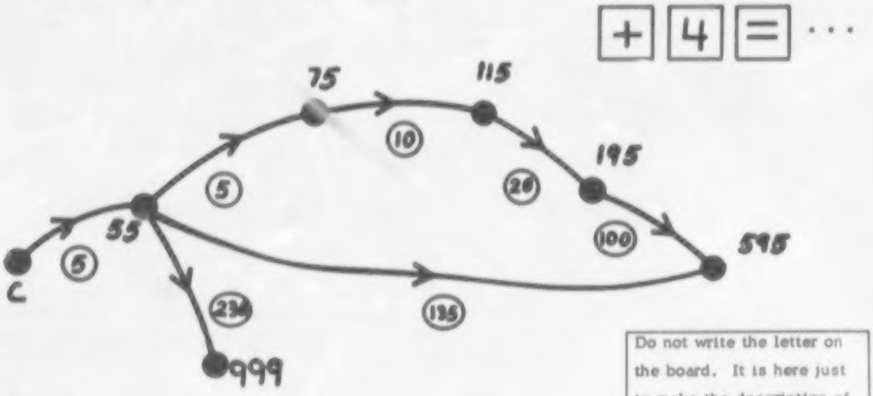
Label the dot "999".

T: How many times must $\boxed{=}$ be pressed to get 999? Why?

S: 236 times; 236 x 4 = 944 and 55 + 944 = 999.

S: Using the arrow picture, I know we get 595 after pressing $\boxed{=}$ 135 times. So we get 599 after pressing $\boxed{=}$ 136 times, and finally we get 999 after pressing $\boxed{=}$ 236 times.

Add this arrow from c to 55 to your picture.



Do not write the letter on the board. It is here just to make the description of the lesson easier to follow.

80

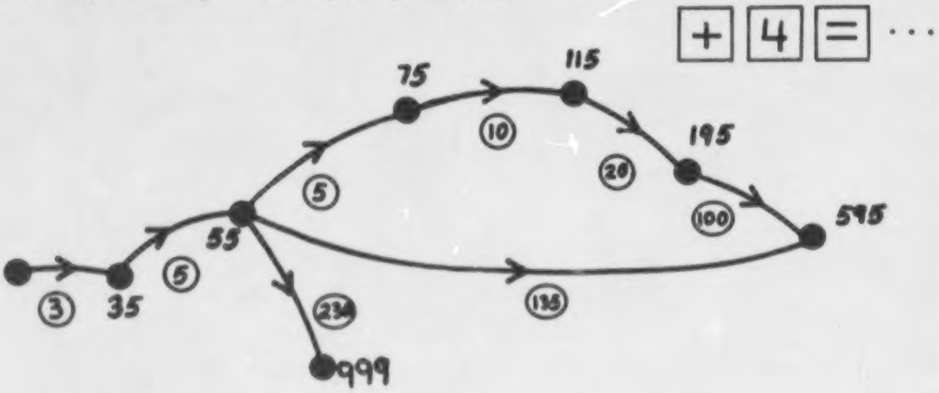
T (pointing to c): From this number I can get 55 by pressing $\boxed{+}$ $\boxed{4}$ and then $\boxed{=}$ five times. What number is this?

S: 35.

T: How did you get 35?

S: 5 × 4 = 20 and 55 - 20 = 35.

Add this arrow ending at 35 to your picture.

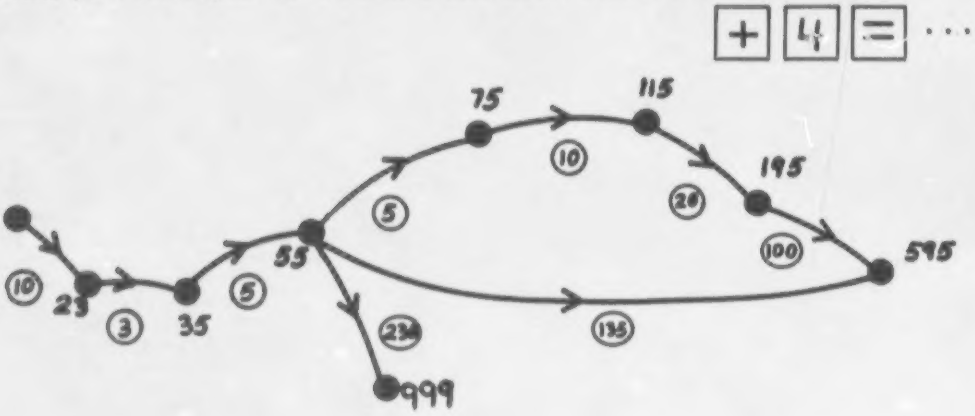


+ 4 = ...
75
115
195
595
55
35
999
③
⑤
⑤
⑩
⑳
(100)
(135)
(234)

T (pointing to the unlabeled dot): What number is here?

S: 23, because 3 × 4 = 12 and 35 - 12 = 23.

Extend your arrow picture to include this arrow ending at 23.



+ 4 = ...
75
115
195
595
55
35
23
999
⑩
③
⑤
⑤
⑩
⑳
(100)
(135)
(234)

26 N

T: What number is here?

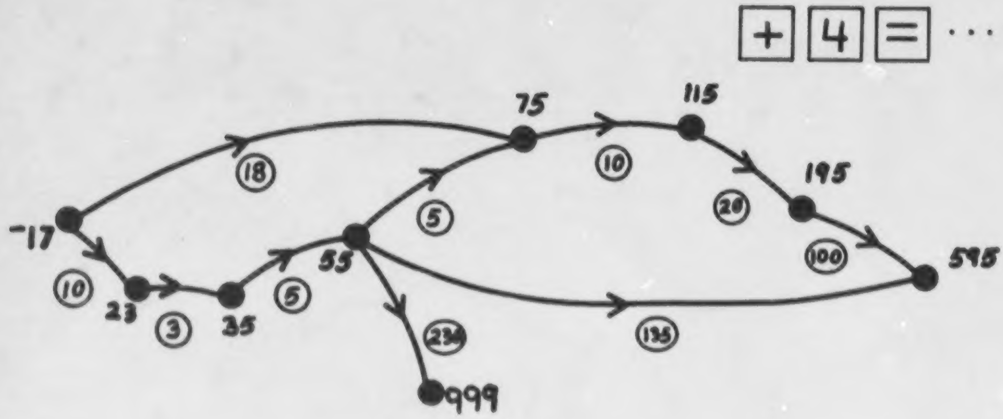
S: ¯17; 10 x 4 = 40 and 23 - 40 = ¯17.

Draw an arrow starting from ¯17 and ending at 55. Put a "○" near the arrow.

T: What should we write in the "○"?

S: "18", because 10 + 3 + 5 = 18.

Fill in the "○".



+
4
=
...
¯17
⑱
⑩
23
③
35
⑤
55
⑤
75
⑩
115
⑳
195
100
595
135
234
999

T: Could I draw a red arrow from 55 to 700? If we cannot, what is the closest number to 700 that could be an ending number for a red arrow starting at 55?

Let the students have a minute to consider the problem.

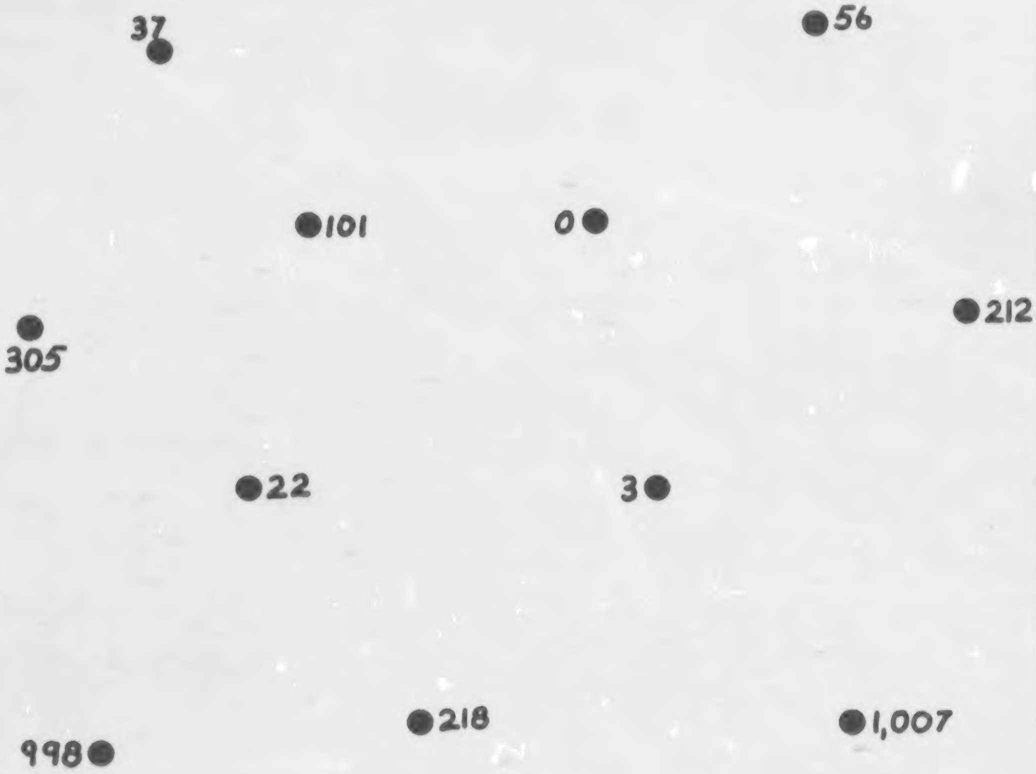
S: 699 is the closest; then 703; but not 700.

Exercise 2

NOTE: You should prepare this drawing ahead of time on the chalkboard or on an easel size sheet of paper which can be taped to the board. A careful placement of the dots can enhance the arrow picture that will be drawn.

Draw this picture on the board and ask the students to copy it.

$\boxed{+}\boxed{4}\boxed{=}\cdots$



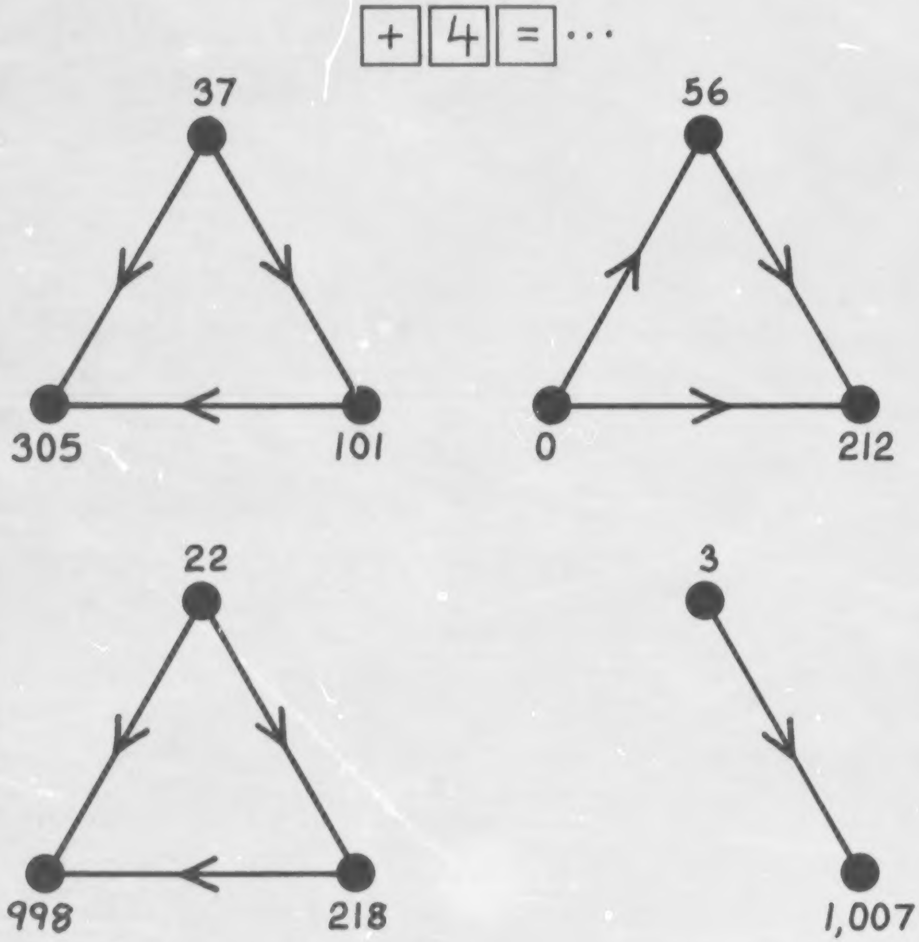
T: Draw some red arrows in your picture. Try to find all the red arrows that are possible. If you wish, use your calculator to help you.

26 N

To be certain everyone understands the problem, ask a student to draw one arrow at the board. Let the students work independently on this for a few minutes. Then ask:

T: Who can draw another arrow in our picture?

Let several students assist in completing the arrow picture at the board. A completed picture is shown below.

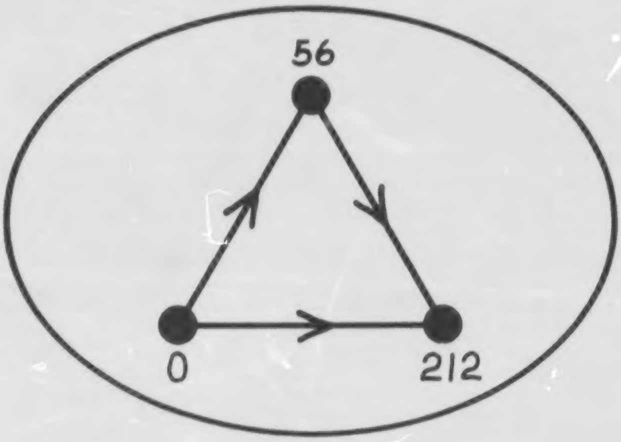


T: Do you notice anything interesting about this arrow picture?

S: There are four different pieces.

T: Good. Let's look at this piece.

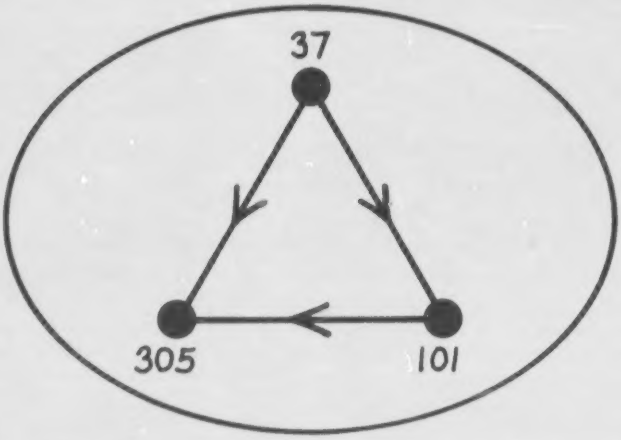
Draw a string around this part of the arrow picture.



T: What do you notice about the numbers in this piece of the picture?

S: They are all multiples of 4.

Draw a string around this part of the arrow picture.



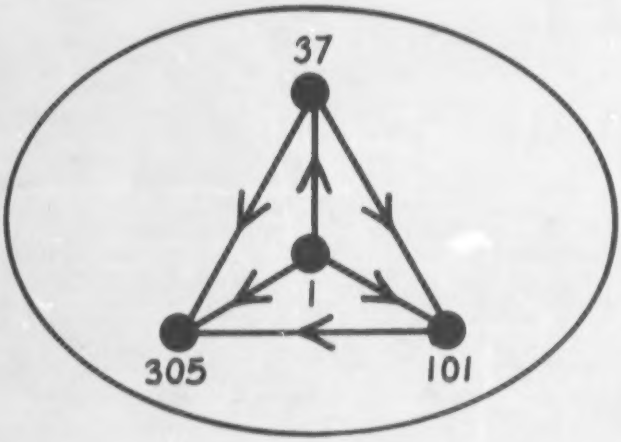
T: What is the smallest whole number that could be joined to this piece of the arrow picture?

S: 1.

85

26 N

Add 1 to your picture.



T: Do you notice anything interesting about these numbers?

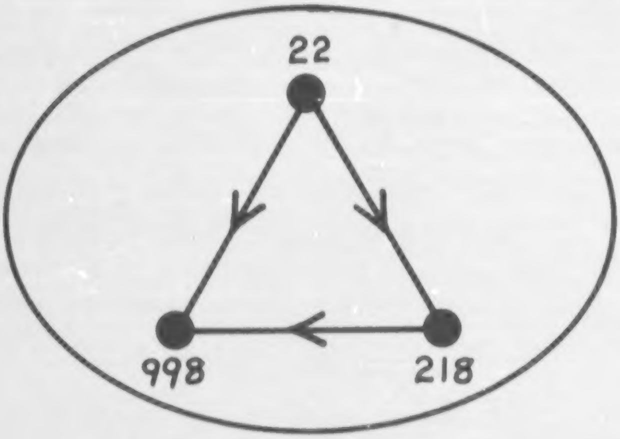
S: Each number is one more than a multiple of 4. 37 = 36 + 1 and 36 is a multiple of 4; 101 = 100 + 1 and 100 is a multiple of 4; and 305 = 304 + 1 and 304 is a multiple of 4.

If this observation is not suggested by a student, suggest it yourself.

T: What could we call these numbers?

S: 1 plus a multiple of 4 (or multiples of 4 plus 1).

Direct the students' attention to the lower left piece of the arrow picture.



T: What is the smallest whole number that could be joined to this piece of the arrow picture?

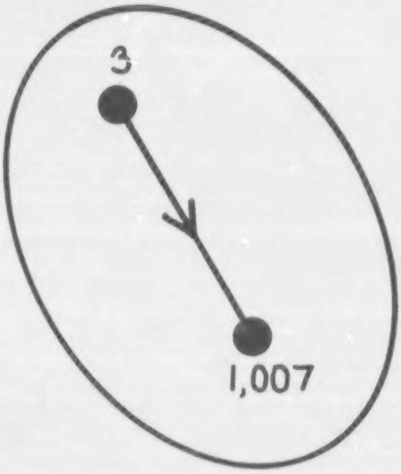
S: 2.

Be sure to add 2 to your picture.

T: What do you notice about these numbers?

S: They are multiples of 4 plus 2.

T: Good. For this last piece, what is the smallest whole number that could be joined to it?



S: 3, but it's already in the picture.

S: The numbers are multiples of 4 plus 3.

Worksheets N13, N14, N15, and N16 are available.

Name

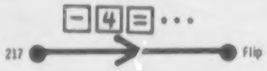
N 13 *

First clue

Flip can be put on this Minicomputer board using exactly two positive checkers.

Flip is 2 or 3 or 4 or 5 or 6 or 8 or 9 or 10 or 12 or 16.

Second clue



Flip is 5 or 9.

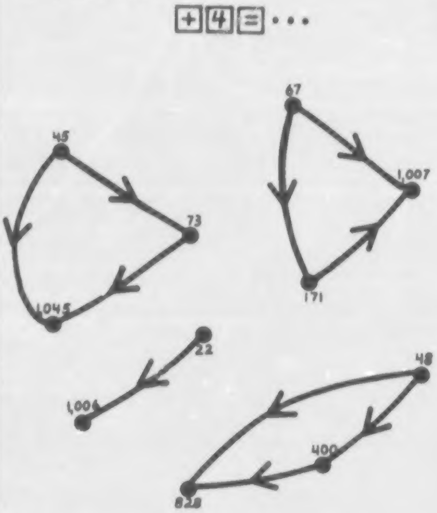
Third clue

Flip is a prime number. Flip is 5.

Name

N 14 **

Draw all the missing arrows.

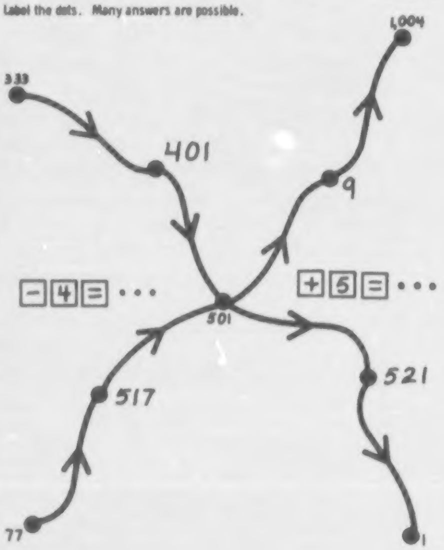


+ 4 = …
67
45
1,007
73
171
1,045
22
1,006
48
400
828

Name

N 15 ***

Label the dots. Many answers are possible.



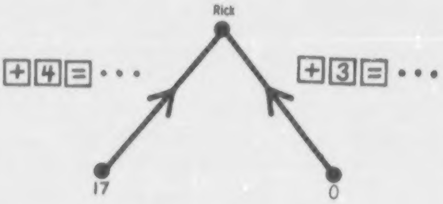
1,004
333
401
9
− 4 = …
+ 5 = …
501
517
521
77
1

(Many solutions are possible.)

Name

N 16 ****

First clue



Rick could be 21; 33; 45; 57; 69; 81; 93; 105; 117; 129; 141; 153; 174; 186; and so on.

Second clue

Rick is between 1,200 and 1,220.

Rick is 1,209

3

COMBINATORICS

"Combinatorics" is a term meaning "the process of (systematically) counting the number of possible ways or outcomes in certain situations". In this chapter, several situations are presented in which it is of interest to "count the possibilities". These situations include:

- counting how many numbers can be obtained on the display of a hand-calculator by pressing, for example,

  $$\boxed{7}\ \boxed{}\ \boxed{2}\ \boxed{}\ \boxed{10}\ \boxed{=},$$

  where one of the four operation symbols: +, −, ×, ÷, is hidden in each blank box (and variations of this situation);

- counting how many secret messages can be sent by writing the messages in "code 3" (i.e., as a number on an abacus with the rule that three checkers on a board represents the same number as one checker on the next board to the left), and variations of this activity using "code 5";

- counting how many permutations there are of a set by using the relation "I hold your left hand with my right hand" on a set of students.

- counting how many different colorings there are of a picture of four squares using red or blue for each square, and also of a picture of nine squares.

In each of these situations a procedure is found for "counting the possibilities".

CAPSULE LESSON SUMMARIES

Section 3.1: Using tree diagrams, find how many different numbers can be put on the display of a hand-calculator by pressing, for example, $\boxed{8}\,\boxed{}\,\boxed{3}\,\boxed{}\,\boxed{9}\,\boxed{=}$, where one of the operation symbols: +, −, x, ÷, is hidden in each blank box. Also, find how many numbers are possible if the symbols in the boxes must be different.
(Worksheets N17, N18, N19)

Section 3.2: Put numbers on an abacus with the rule that three checkers on a board represents the same number as one checker on the next board to the left. Relate configurations of checkers on this abacus to messages in a story about six spies observing three bridges. Using this Base Three abacus, determine how many messages can be written in "code 3".
(Worksheets N20, N21, N22, N23)

Section 3.3: Put numbers on an abacus with the rule that five checkers on a board represents the same number as one checker on the next board to the left, and determine how many messages in "code 5" can be sent with numbers on this Base Five abacus.
(Worksheet N24)

Section 3.4: Find the number of arrow pictures that can be drawn with dots for eight people and arrows for "I hold your left hand with my right hand".

Section 3.5: Color each square of the 2 by 2 picture with either red or blue, and determine how many such pieces of "random art" are possible.
(Worksheets N25, N26, N27)

Section 3.6: Color each square of the 3 by 3 picture with either red or blue, and find how many different pictures are possible. Determine which pictures are "one-of-a-kind", "two-of-a-kind", etc.

3.1 Hand-Calculator Sentences

NOTE: Students do not need to have hand-calculators for this lesson; however, they will have to think about how a hand-calculator operates. We assume that the hand-calculators being discussed do chain operations from left to right (see the General Introduction, Section 2(f) for a more detailed commentary on hand-calculators).

Exercise 1

Put this expression on the board.

$$\boxed{7}\boxed{+}\boxed{2}\boxed{\times}\boxed{10}\boxed{=}\boxed{}$$

T: Today we will pretend we are using a hand-calculator. You must think about what it does, because we will not really use one. This expression is a hand-calculator sentence. Can you read it?

S: Press $\boxed{7}$ $\boxed{+}$ $\boxed{2}$ $\boxed{\times}$ $\boxed{10}$ $\boxed{=}$ and the number on the display goes in the blank box.

T: Very good. We start by putting 7 on the display; press $\boxed{7}$.

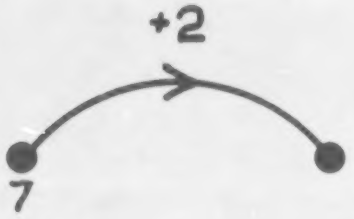
Start an arrow picture by drawing a dot and labeling it "7".

T: Then we press $\boxed{+}$ $\boxed{2}$. What does the calculator do?

S: It adds 2.

3 N

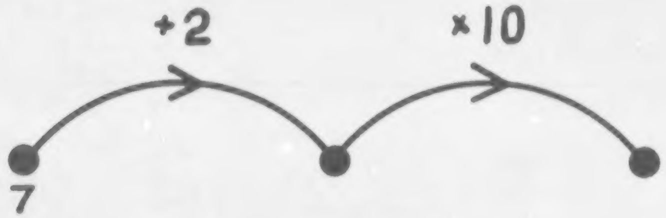
In your picture, draw a + 2 arrow starting at 7.



T: Next we press $\boxed{\times}$ $\boxed{10}$. What does the calculator do?

S: It multiples by 10.

Extend your picture with a × 10 arrow.

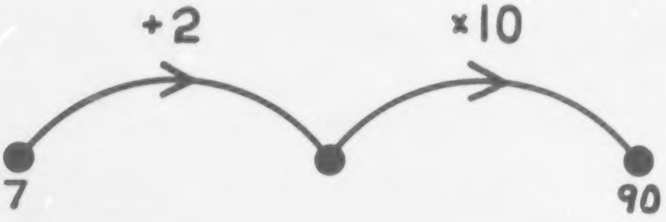


T: Then when we press $\boxed{=}$, and this number (point to the dot at the end of the × 10 arrow) appears on the display. What number is here?

S: 90, because 7 + 2 = 9 and 9 × 10 = 90.

Label the dot and write "90" in the blank box of the hand-calculator sentence.

$\boxed{7}\ \boxed{+}\ \boxed{2}\ \boxed{\times}\ \boxed{10}\ \boxed{=}\ \boxed{90}$



NOTE: In usual notation, parentheses would be used to indicate the order in which the operations are performed. The previous hand-calculator sentence would be written:

$$(7 + 2) \times 10 = 90.$$

Erase the arrow picture and change the hand-calculator sentence as shown below.

$$\boxed{7}\boxed{}\boxed{2}\boxed{}\boxed{10}\boxed{=}\boxed{ZA}$$

T: Za is a secret number. Each blank box hides one of the symbols: + , − , × , ÷. Both blank boxes could hide the same symbol. What numbers could Za be?

S: 90. If "+" were in the first box and "×" were in the second box, then 90 would appear on the display. $[(7 + 2) \times 10 = 90]$

S: 4. If "×" were in the first box and "−" were in the second box, then Za would be 4. $[(7 \times 2) - 10 = 4]$

S: 19. If "+" were in both boxes, Za would be 19. $[(7 + 2) + 10 = 19]$

S: 13.5. If "÷" were in the first box and "+" were in the second box, Za would be 13.5. $[(7 \div 2) + 10 = 13.5]$

The examples above are a few of the many responses that might be given by your students. Let several students answer until it is obvious that most students understand the problem.

T: How many ways are there to complete this hand-calculator sentence using the four symbols: + , − , × , ÷? Remember, we can use the same symbol twice.

S: 16.

T: Why?

S: Any one of the four symbols can be put in the first box. Then any one of the four symbols can be put in the second box. So there are 4 x 4 or 16 ways to complete this hand-calculator sentence.

Exercise 2

Put this incomplete hand-calculator sentence on the board.

$$\boxed{27}\,\boxed{}\,\boxed{3}\,\boxed{}\,\boxed{2}\,\boxed{}\,\boxed{10}\,\boxed{=}\,\boxed{\text{Clo}}$$

NOTE: The usual notation using parentheses for this hand-calculator sentence is

$$((27\,\boxed{}\,3)\,\boxed{}\,2)\,\boxed{}\,10 = \text{Clo}$$

The operations are performed from left to right.

T: Each blank box hides one of the symbols: +, −, x, ÷. Clo is a secret number. What numbers could Clo be?

S: If "+" were in the first box, "−" in the second box, and "x" in the third box, then Clo would be 280. ((27 + 3) − 2) x 10 = 280)

Record student responses on the board below the hand-calculator sentence.

$$\boxed{27}\,\boxed{}\,\boxed{3}\,\boxed{}\,\boxed{2}\,\boxed{}\,\boxed{10}\,\boxed{=}\,\boxed{\text{Clo}}$$

$$+ \qquad - \qquad \times \longrightarrow 280$$

Let several students respond orally and record their answers on the board in a manner similar to the above. When it is clear that most students understand the problem, ask them to write four or five more numbers that Clo could be on their paper. Encourage students to use the same recording scheme that you used on the board.

T: How many ways are there to complete this hand-calculator sentence using the four symbols: +, −, ×, ÷? We can use the same symbol more than once.

Let students make guesses, but encourage some students to explain their answers. Try to get a good explanation from those who suggest "64" as this is the correct answer.

If no one gives the correct answer, make the following suggestion.

T: Forget about the third blank box for the moment. Do you remember how many ways there were to complete a hand-calculator sentence when we had only two blank boxes?

S: 16, because 4 × 4 = 16.

T: Now we again consider the third blank box. There are four choices to go with each of the 16 possibilities.

S: 16 × 4 = 64, so there must be 64 ways.

Exercise 3

Put this incomplete hand-calculator sentence on the board.

$$\boxed{100}\ \boxed{}\ \boxed{5}\ \boxed{}\ \boxed{2}\ \boxed{}\ \boxed{10}\ \boxed{=}\ \boxed{\text{Kim}}$$

3N

T: Again, each blank box hides one of the symbols: +, -, x, ÷. But this time I am going to change the rule. You cannot use the same symbol twice in one sentence. On your paper, write some numbers that Kim could be.

If necessary, remind the students with an example about the new rule. For instance, $\boxed{100}\,\boxed{+}\,\boxed{5}\,\boxed{-}\,\boxed{2}\,\boxed{\times}\,\boxed{10}\,\boxed{=}\,\boxed{1{,}030}$ is permitted, but $\boxed{100}\,\boxed{+}\,\boxed{5}\,\boxed{+}\,\boxed{2}\,\boxed{\times}\,\boxed{10}\,\boxed{=}\,\boxed{1{,}070}$ is not permitted since "+" appears twice.

As you walk around observing the students, encourage them to use the recording scheme of the previous exercise.

$$+ \quad - \quad \times \longrightarrow 1{,}030$$

$$\div \quad \times \quad - \longrightarrow 30$$

and so on.

After a few minutes of work, ask several students to suggest numbers that Kim could be. When you have recorded three or four possibilities, ask:

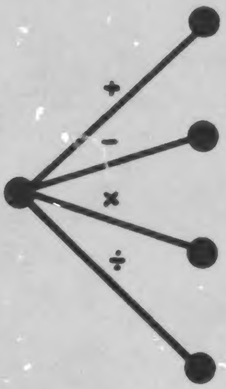
T: How many ways are there to complete this hand-calculator sentence using the four symbols: +, -, x, ÷? Remember, you cannot use the same symbol twice in any one sentence.

Ask students who answer "24", the correct answer, to try to explain their answers.

T: Let us think about this problem. In the first box I can write any one of the four symbols: +, -, x, ÷.

Draw this tree on the board. (Use any other color that has not been used for black.)

$\boxed{100}\ \boxed{\quad}\ \boxed{5}\ \boxed{\quad}\ \boxed{2}\ \boxed{\quad}\ \boxed{10}\ \boxed{=}\ \boxed{\text{Kim}}$

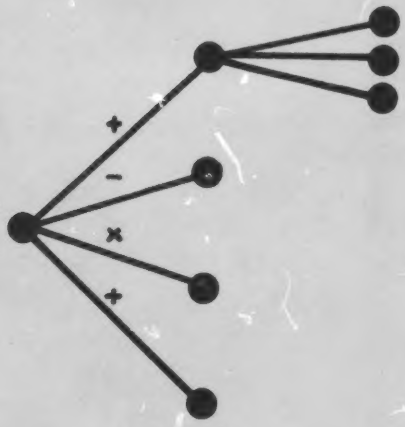


T: Suppose I choose "+ " for the first box. (Trace the red branch.) Which symbol could be written in the second box?

S: "-", "x", or "÷" but not "+ " since it has already been used once.

Add this information to your tree.

$\boxed{100}\ \boxed{\quad}\ \boxed{5}\ \boxed{\quad}\ \boxed{2}\ \boxed{\quad}\ \boxed{10}\ \boxed{=}\ \boxed{\text{Kim}}$



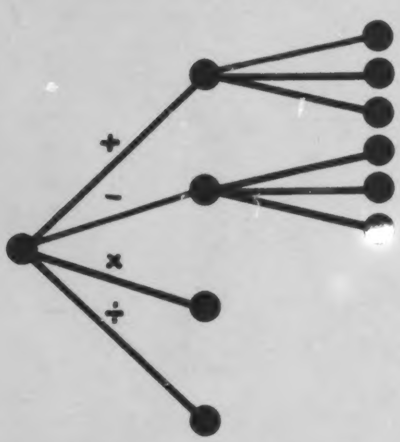
3 N

T: Suppose "-" is chosen for the first box. (Trace the blue branch.) Which symbol could be written in the second box?

S: "+", "x", or "÷" could be written in the second box.

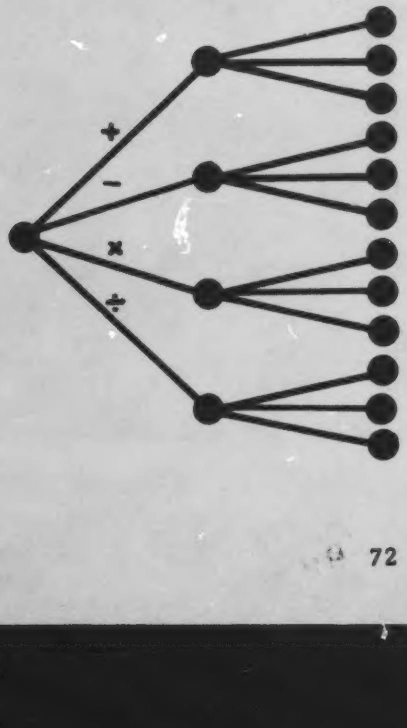
Expand your tree to show these possibilities.

100 ☐ 5 ☐ 2 ☐ 10 = Kim



Continue in this manner expanding the "second level" on the tree. Upon completion of the second level, your tree should look something like this.

100 ☐ 5 ☐ 2 ☐ 10 = Kim



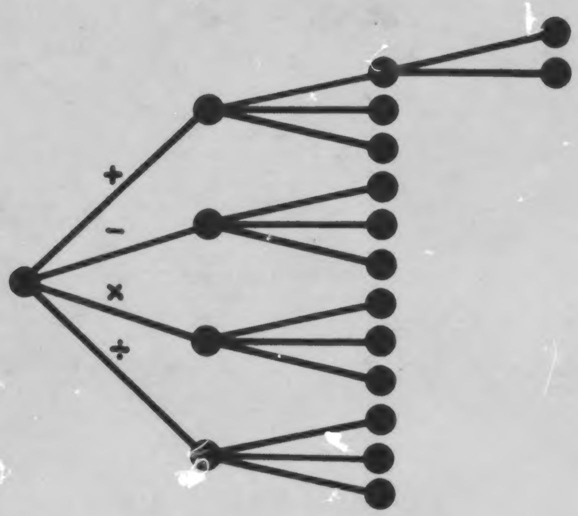
Start the "third level" but do not complete it.

T: Suppose "+ " has been chosen for the first box and "-" for the second box. (Trace the red-blue path.) What choice do I have for the third box?

S: "x" or "÷".

Expand your tree to include these two choices.

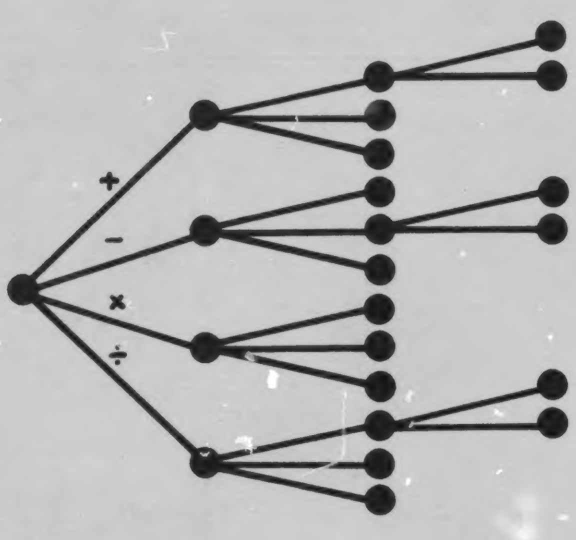
100 □ 5 □ 2 □ 10 = KiM



At this point, ask a student to trace a path, to tell what operations are used, and to add the branches for the two choices remaining for the third box. It is not necessary to complete the tree. The next illustration shows what your tree might look like after three students have traced paths.

3N

$\boxed{100}\ \boxed{}\ \boxed{5}\ \boxed{}\ \boxed{2}\ \boxed{}\ \boxed{10}\ \boxed{=}\ \boxed{\text{Kim}}$



T: Each path of this tree represents a way to complete this hand-calculator sentence and each completed hand-calculator sentence can be represented by a path. If we were to finish the tree, how many paths would there be?

S: 24 paths.

T: Do you know a way to calculate the number of paths without counting?

S: 4 x 3 x 2 = 24.

T: Very good. There are 24 ways to complete our hand-calculator sentence.

Worksheets N17, N18, and N19 are available.

Name ______________________

N 17 *

Jo is a secret number.

First clue

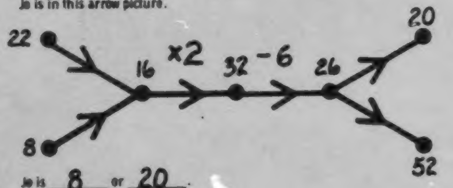
One of the symbols: + , − , x is hidden in each blank of this hand-calculator sentence. A symbol may be used twice.

$\boxed{7}\,\boxed{}\,\boxed{3}\,\boxed{}\,\boxed{2}\,\boxed{=}\,\boxed{\text{Jo}}$

Jo is 12 or 8 or 20 or 6 or 2 or 42 or 23 or 19.

Second clue

Jo is in this arrow picture.



Jo is 8 or 20.

Third clue

If you add 1 to Jo, you will get a square number.

Jo is 8.

Name ______________________

N 18 **

Lou is a secret number.

First clue

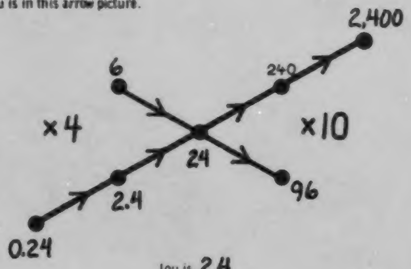
One of the symbols: + , x , ÷ is hidden in each blank of this hand-calculator sentence. Each symbol may be used only once.

$\boxed{9}\,\boxed{}\,\boxed{3}\,\boxed{}\,\boxed{2}\,\boxed{}\,\boxed{10}\,\boxed{=}\,\boxed{\text{Lou}}$

Lou is 2.4 or 60 or 2.9 or 23.5 or 50 or 16.

Second clue

Lou is in this arrow picture.



Lou is 2.4.

Name ______________________

N 19 ***

Kir is a secret number.

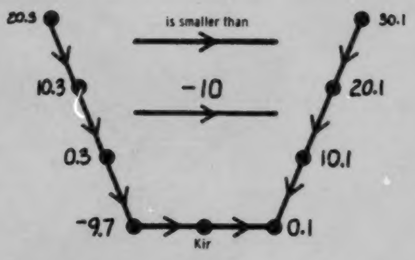
First clue

One of the symbols: + , ÷ is hidden in each blank of this hand-calculator sentence. The same symbol may be used in all three blanks.

$\boxed{8}\,\boxed{}\,\boxed{2}\,\boxed{}\,\boxed{10}\,\boxed{}\,\boxed{10}\,\boxed{=}\,\boxed{\text{Kir}}$

Kir is 30 or 0.04 or 2 or 24 or 11 or 10.4 or 1.4 or 0.1.

Second clue



Kir is 0.04

7N

3.2 Spies and Bridges #1

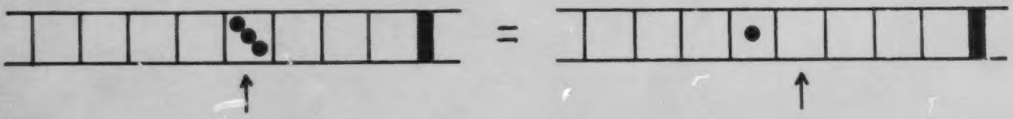
Exercise 1

Draw this part of an abacus on the board or use Minicomputer boards.[*]

T: Do you remember the binary abacus and its rule?

S: Yes, two checkers on one board represents the same number as one checker on the next board to the left.

T: Good. Today we will change the rule so that we have a Base Three abacus. The new rule is:



> Three checkers on a board represents the same number as one checker on the next board to the left.

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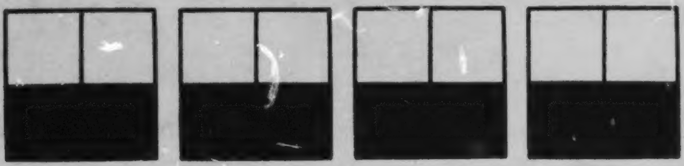
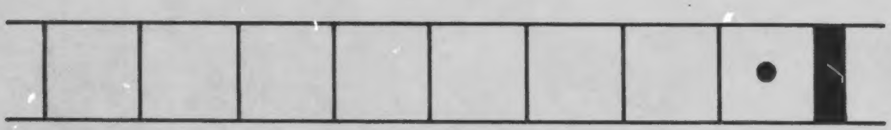
[*] If your chalkboard is not magnetic, a magnetic abacus can be made using your demonstration Minicomputer boards. Cover the red and white squares with paper and use only the brown and purple squares on each board.

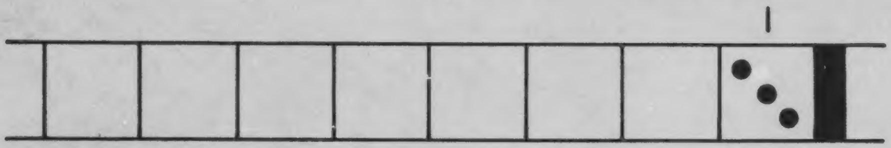


Put this checker on the abacus.



T: This is the number 1.

Label the square "1" and put two more checkers on the ones' board.

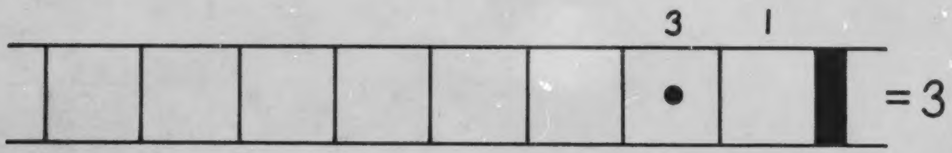


T: What number is this?

S: 3.

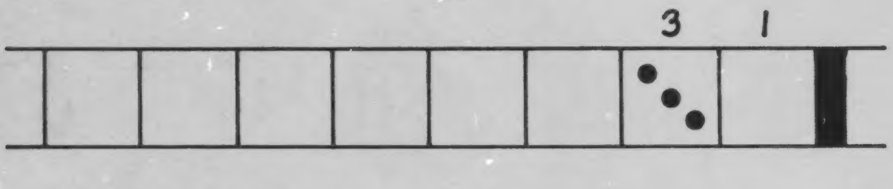
T: Can you put 3 on the abacus using exactly one checker?

Let a student make the play which puts 3 on the abacus with one checker, and then label the board "3".



7N

Put two more checkers on the threes' board.

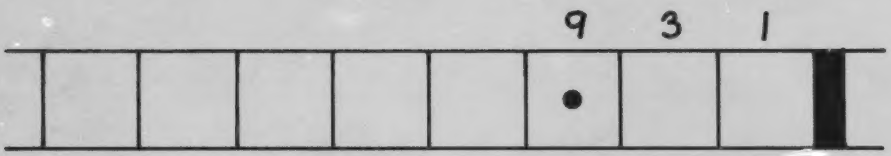


T: What number is this?

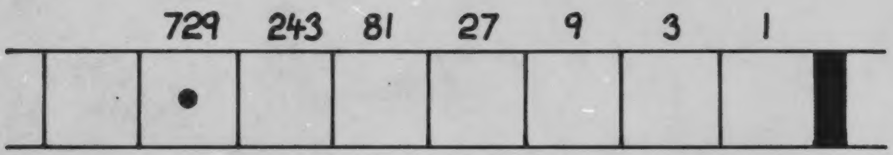
S: 9.

T: Can you show 9 using exactly one checker?

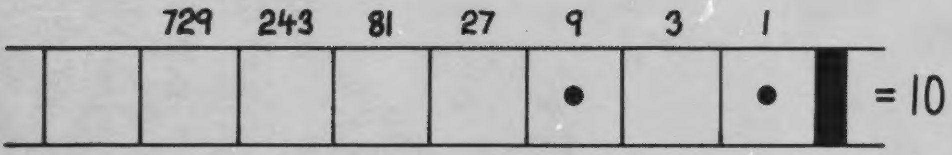
Let a student make the play to show 9 with one checker and label the board.



Repeat this activity until you have at least seven boards of the abacus labeled.


729
243
81
27
9
3
1

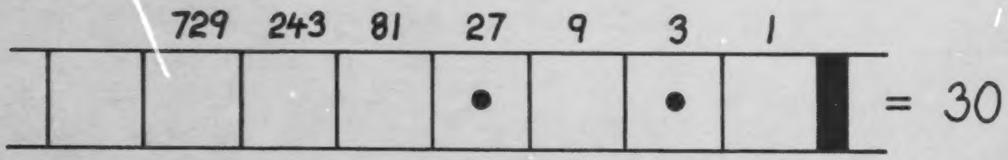
T: Put 10 on the abacus using at most two checkers on a board.


729
243
81
27
9
3
1
= 10

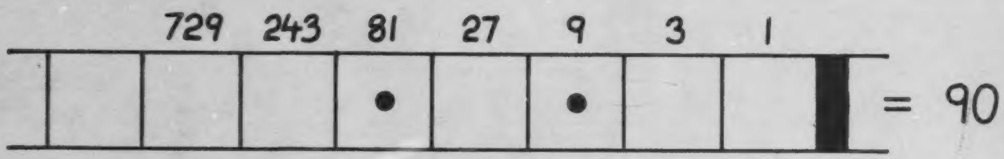
T: Put on 30 using at most two checkers on a board.

Encourage your students to observe:

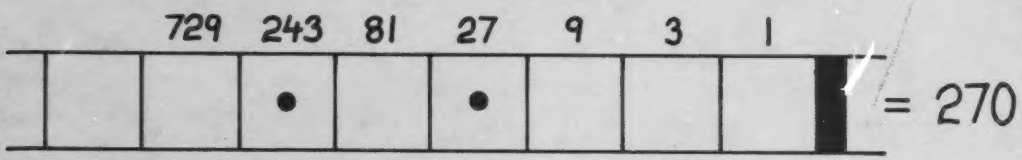
S: We multiply by three, so we just move each checker one board to the left.


729
243
81
27
9
3
1
= 30

T: Now put 90 on the abacus.


729
243
81
27
9
3
1
= 90

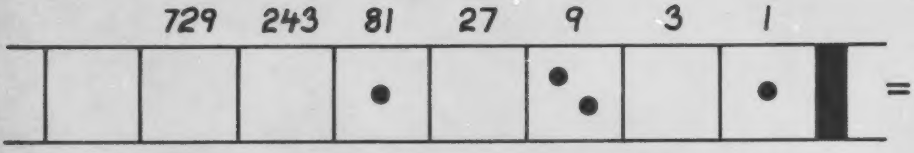
T: Now 270.


729
243
81
27
9
3
1
= 270

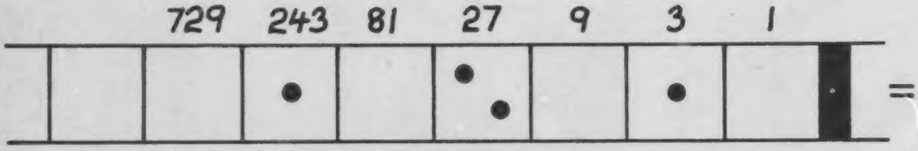
T: Draw an abacus on your paper and put on 100. Use at most two checkers on a board. Then draw another abacus below the first and put on 300.

Invite students to put 100 and 300 on the demonstration abacus.

7 N

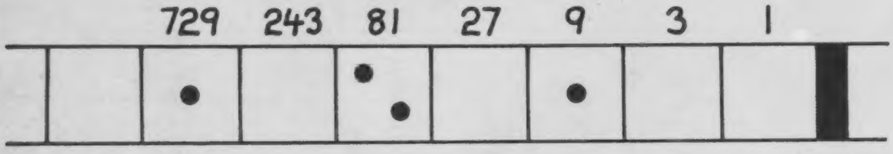

729
243
81
27
9
3
1

= 100


729
243
81
27
9
3
1

= 300

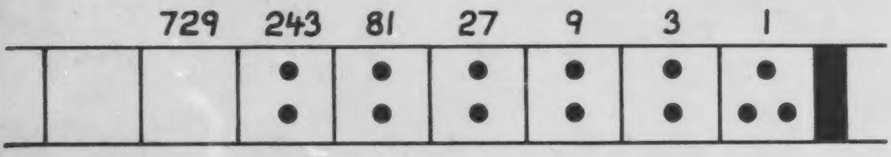
Put this configuration on the abacus.


729
243
81
27
9
3
1

T: What number is this?

S: 900. You moved each checker one board to the left, so we just multiply by 3; that is, 3 x 300 = 900.

Put this configuration on the abacus.


729
243
81
27
9
3
1

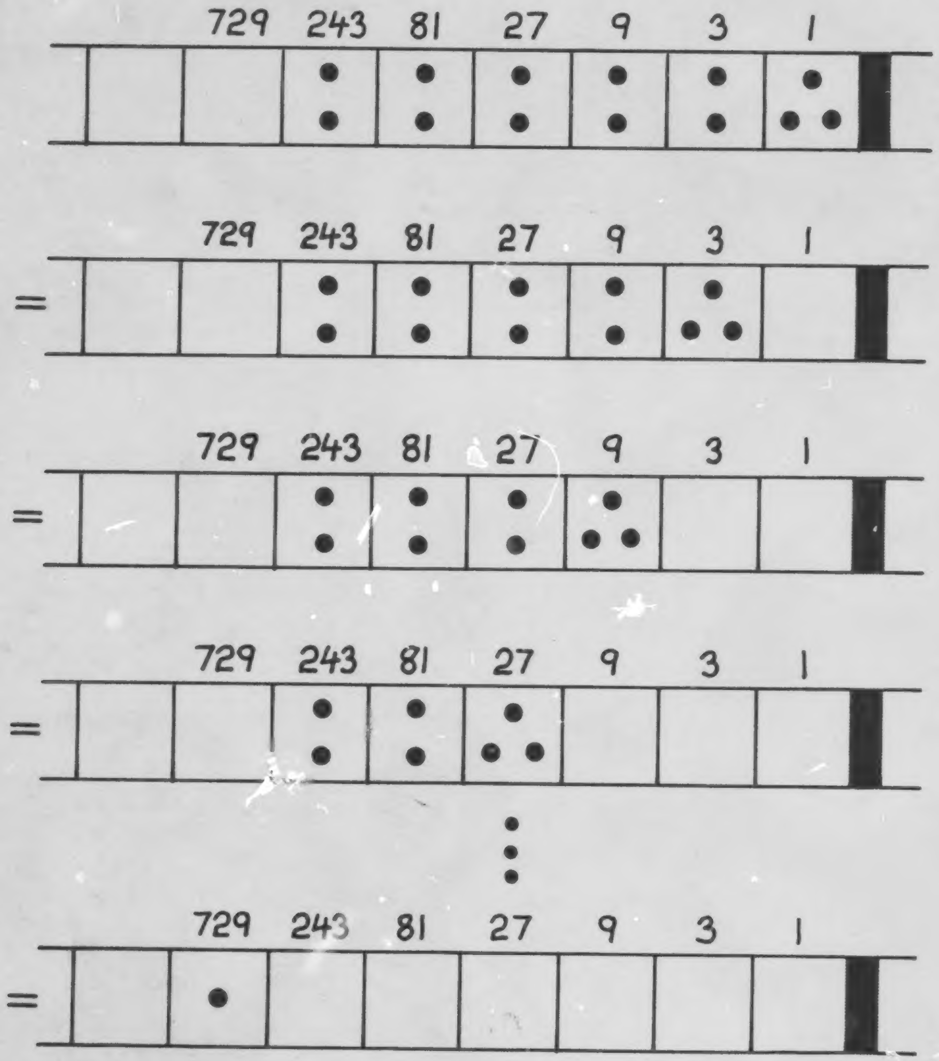
T: What number is this?

Many students will do all the indicated additions to find this number, but others will find a short cut. When it is obvious that a short cut has been found, ask the students how they arrived at the number.

S: The number is 729.

T: Can you show us how you arrived at 729?

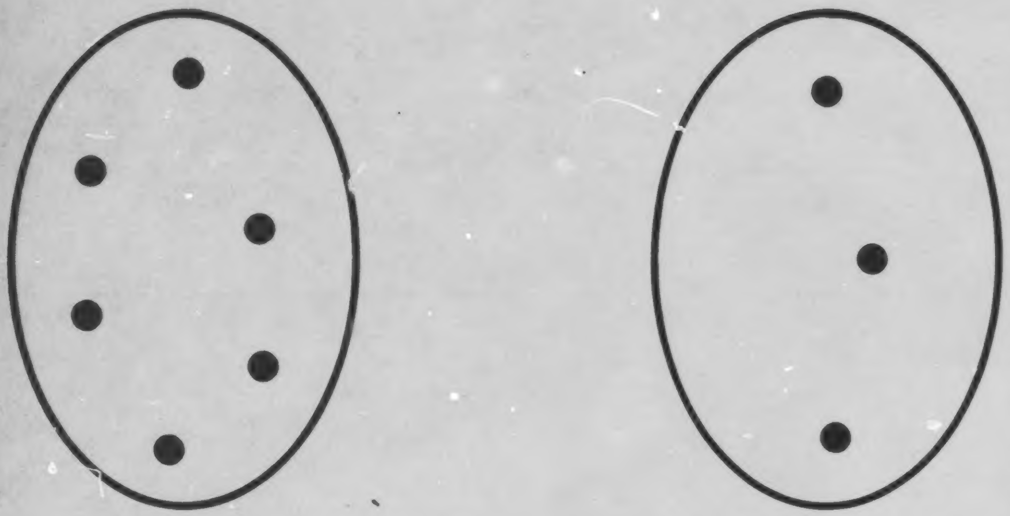
Let a student who has found a short cut demonstrate using the abacus and making the following sequence of plays.



7N

Exercise 2

Draw these strings and dots on the board.



T: This is a story about a spy named "Boris". Boris has six helpers whose code names are "a", "b", "c", "d", "e", and "f".

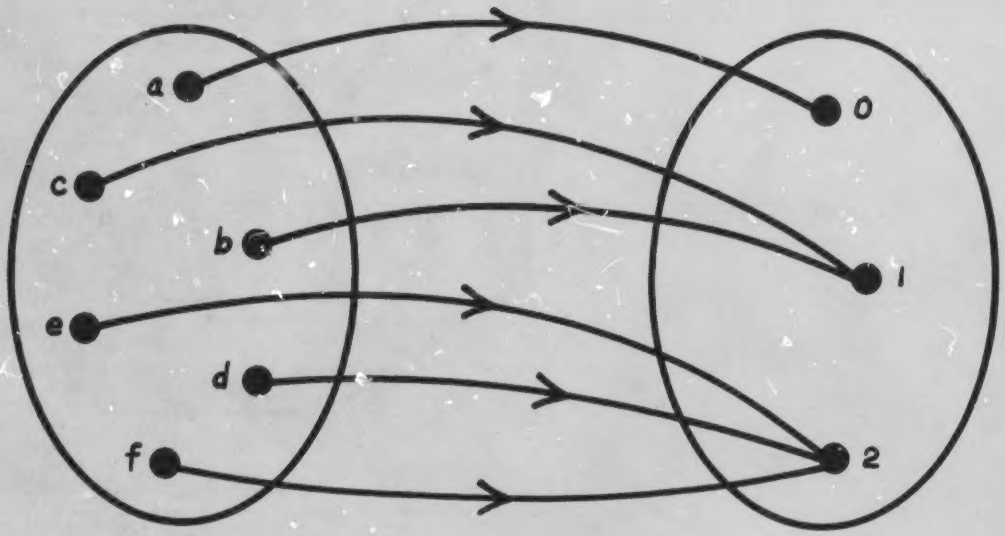
Label the six dots in the string at the left. The order in which you label the dots is of no consequence.

T: Each day Boris' job is to assign his six helpers to observe three bridges. We will call the bridges "0", "1", and "2".

Label the three dots in the string at the right "0", "1", and "2" in any order.

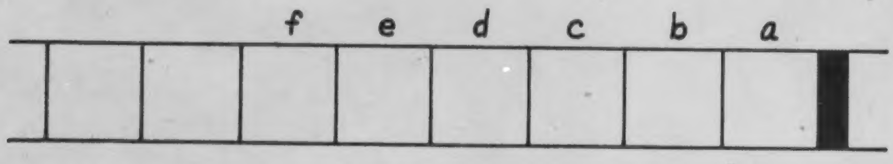
T: Boris might assign his helpers to observe only one bridge or two bridges or all three bridges. He uses an arrow picture to record how the helpers are assigned to the bridges. One day, Boris makes this assignment.

Add these arrows to your picture.



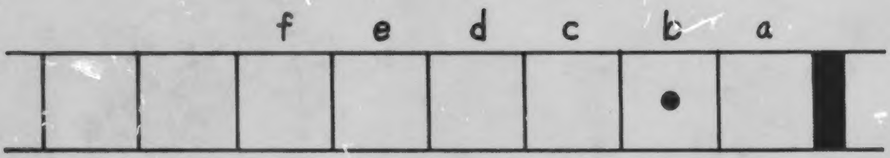
T:	Which bridge does c observe?
S:	1.
T:	Who observes bridge number 2?
S:	d, e, and f.
T:	Each day Boris must transmit this assignment to headquarters. He must send a secret message, but the arrow picture is certainly not very secret. Since there are three bridges, Boris decided that he could use the Base Three abacus. How could he use the abacus?

Allow a few minutes for discussion. The students should have several ideas about how the message could be coded. It is likely that someone will suggest that the boards of the abacus be labeled with the names of the helpers.

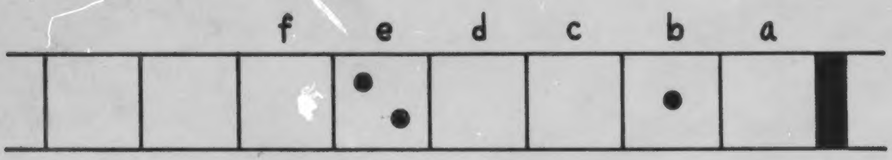


T: A very good idea! How can we show that b observes bridge number 1?

Invite a student to show this.

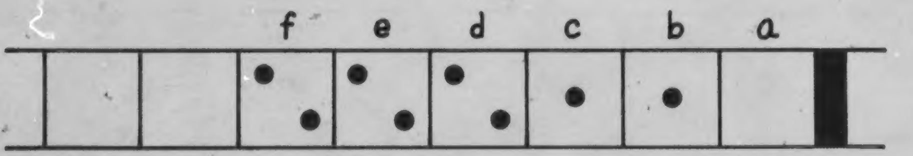


T: How can we show that e observes bridge number 2?



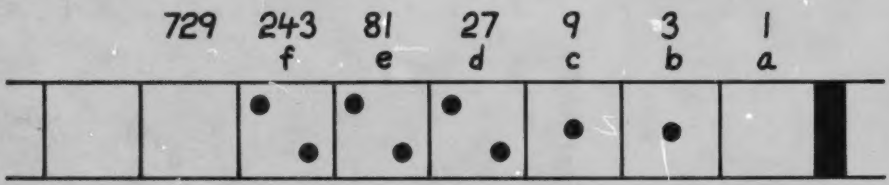
T: Now it is possible to translate the arrow picture to the abacus.

Ask a few students to finish putting checkers on the abacus. When complete, this configuration should be on the abacus.



T: Boris thought that this message was still too easy to decode, so he decided to send the number that is on the abacus. What number is on the abacus?

To make the problem easier, add these labels to the abacus.



S: The number is 714.

T: Boris sent the message "714", but he added something so that headquarters could decode it and would know they were observing three bridges.

Write this on the board.

714
code 3

Erase the arrows in your arrow picture.

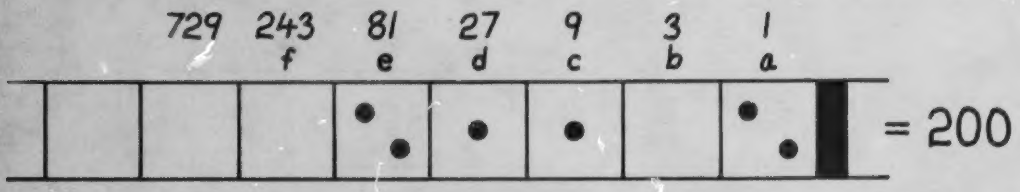
T: On another day, Boris sent this message.

200
code 3

Decode this message and draw the arrow picture. First, draw the Base Three abacus and put on 200.

7 N

Ask a student to put 200 on the demonstration abacus.


729
243
f
81
e
27
d
9
c
3
b
1
a
= 200

T: Good. Which bridge does b observe?

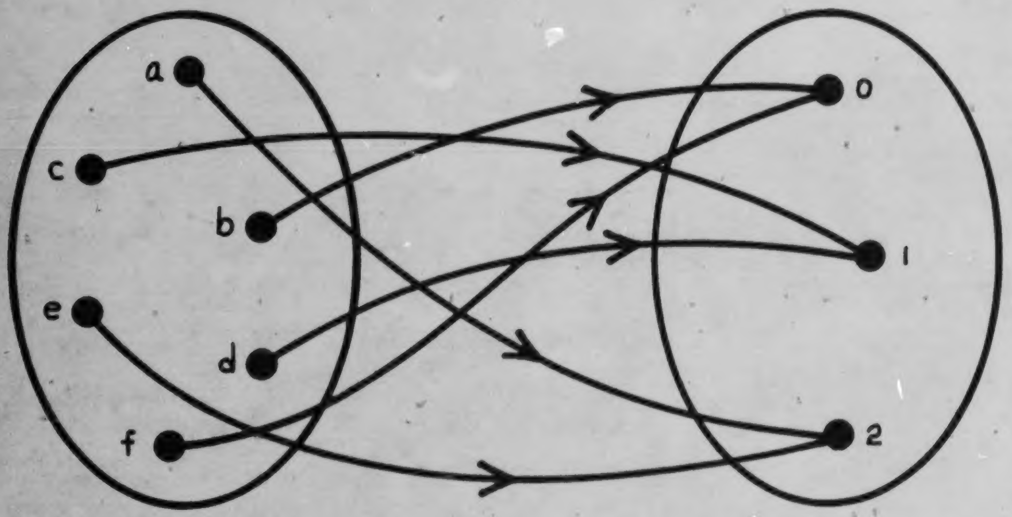
S: Bridge number 0.

Let a student draw an arrow from b to 0.

T: Which bridge does e observe?

S: Bridge number 2.

Let a student draw an arrow from e to 2. Continue in this way until the arrow picture is complete.

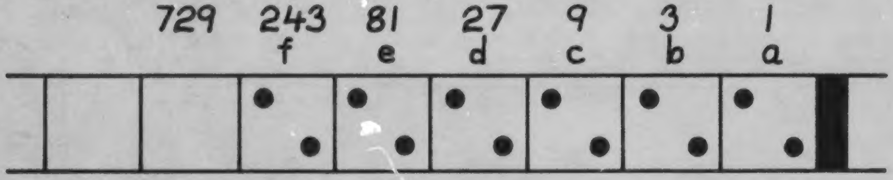


T: How many different messages do you think boris can send using this code?

Accept any estimate.

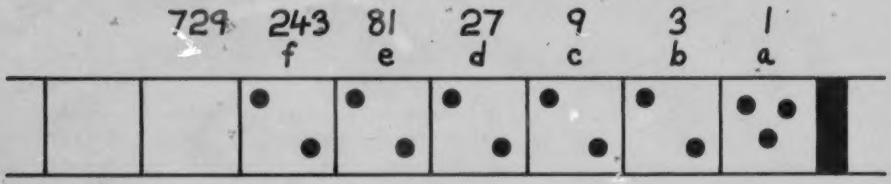
T: Which is the largest number Boris can send?

Invite a student to put this number on the abacus.



T: What number is this?

A quick way, which you could suggest to the class, to calculate this number is to put an extra checker on the ones' board.



In Exercise 1, this number was found to be 729. Since an extra checker was added, the largest number must be 729 - 1 = 728.

T: Which is the smallest number Boris can send?

S: 0.

T: For each number there is a message and for each message there is a number. How many numbers from 0 to 728?

S: 729.

T: How many messages can Boris send?

S: 729.

T: Good. Boris can send 729 messages.

Worksheets N20, N21, N22, and N23 are available.

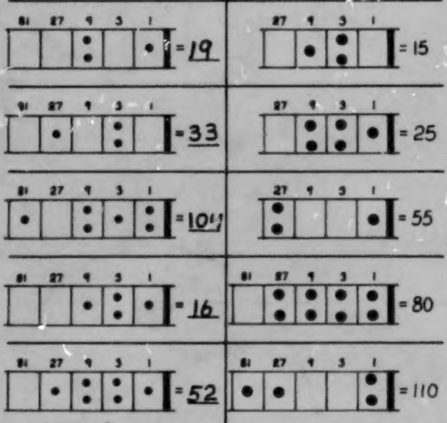
Name ____________________

N 20 *

Base Three Abacus

What number is on the abacus?

Put the number on the abacus. Use at most two checkers on each board.


81 27 9 3 1
27 9 3 1
81 27 9 3 1

= 19 = 15

= 33 = 25

= 104 = 55

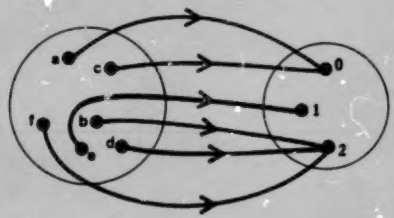
= 16 = 80

= 52 = 110

Name ____________________

N 21 **

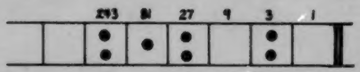
This arrow picture shows how the spy Boris assigned his six helpers to observe three bridges.



Write the secret message that he will send to communicate this information.

627 (Code 3)

Explain your answer below with the help of the Base Three abacus.


243 81 27 9 3 1

= 486 + 81 + 54 + 6 = 627

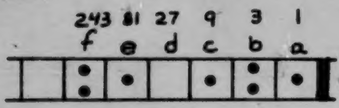
Name ____________________

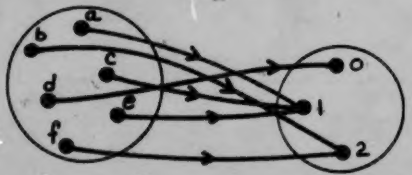
N 22 ***

Boris sent this secret message to headquarters.

583 (Code 3)

Can you decode his message? Use the Base Three abacus and an arrow picture.

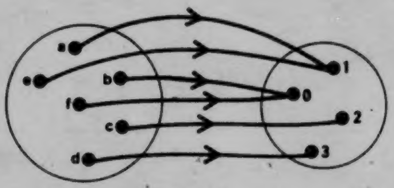

243 81 27 9 3 1
f e d c b a



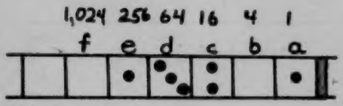
Name ____________________

N 23 ****

Today the spy Boris has FOUR bridges to observe. This arrow picture shows how he assigned his six helpers.


a
e
b
f
c
d
1
0
2
3

Can you invent a new code to send this message secretly? 481 (CODE 4)
Explain it below.


1,024 256 64 16 4 1
f e d c b a

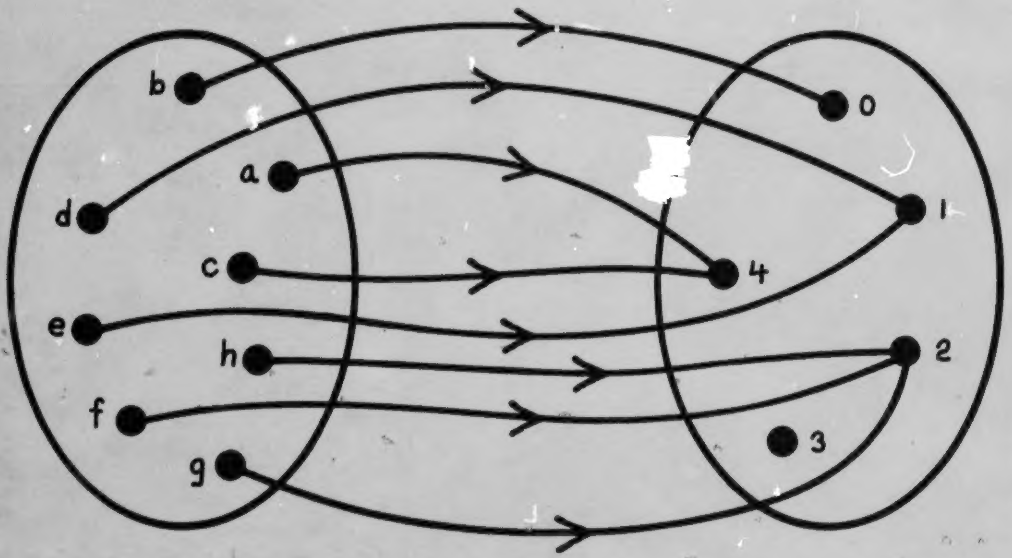
256 + 192 + 32 + 1 = 481

3.3 Spies and Bridges #2

NOTE: This lesson may be a little too long for a single, uninterrupted period; but not so long as to warrant two lessons. If your schedule permits, a five or ten minute break or some other short activity could be inserted between Exercises 1 and 2.

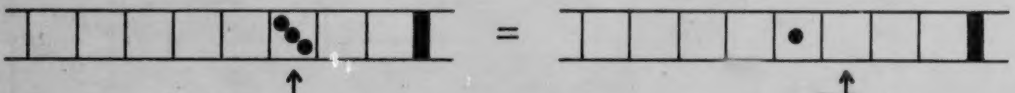
Exercise 1

Draw this arrow picture on the board.



T: Do you remember the story of the spy named "Boris"?

Let the students tell as much of the story as they can recall. Be sure that they or you mention the Base Three abacus rule:



Three checkers on a board represents the same number as one checker on the next board to the left.

T: Today I want to tell you about another spy, Natasha. Natasha has eight helpers whose code names are "a", "b", "c", "d", "e", "f", "g", and "h".

Each day Natasha must assign her eight helpers to observe five bridges. We call the bridges "0", "1", "2", "3", and "4".

Natasha sometimes used arrow pictures to record how the helpers are assigned to the bridges. I have drawn this arrow picture to show how the helpers were assigned on one particular day. Which bridge did c observe?

S: 4.

T: Who observed bridge number 2?

S: f, g, and h.

T: On this particular day, were all the bridges observed?

S: No, 3 was not observed.

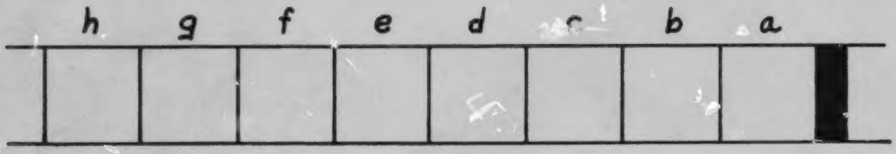
T: Each day Natasha transmits this assignment to headquarters. She thought she might be able to use the code three that Boris invented. What do you think?

Let the class discuss this situation. Someone should suggest that checkers be placed on an abacus.

117

14 N

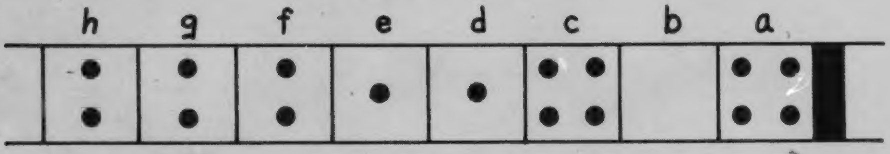
Draw this part of an abacus on the chalk board (or use Minicomputer boards).



T: How could we show that a observed bridge 4?

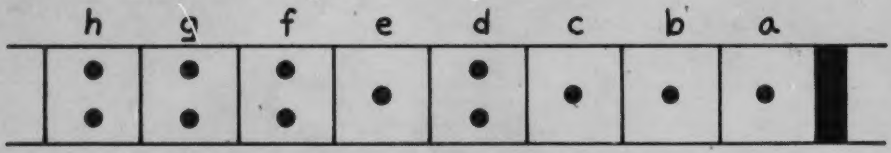
S: Put four checkers on the board labeled "a".

Put these checkers on the abacus and continue in this manner until your abacus has this configuration on it.



T: Do you think we can still use code three for our message?

Again let the class discuss this situation. Some students might suggest that using code three would be confusing because by moving checkers according to the Base Three rule, one can show that this message has the same code number as:



In other words, headquarters would not be able to decode the message accurately. If this is not suggested, mention it yourself. It is not necessary, however, to dwell on this point.

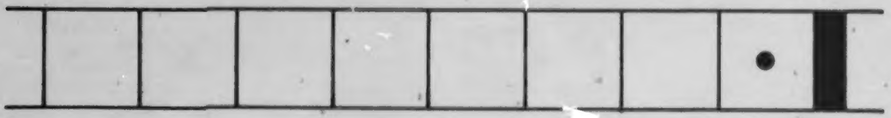
T: Boris used code three because there were only three bridges to observe. On each board he could put 0, 1, or 2, checkers showing a helper's observing bridge number 0, 1, or 2, respectively. And no other message had the same code, because each square had less than three checkers on it.

Now there are five bridges to observe, so Natasha must use a different code. What do you suggest?

S: Code five.

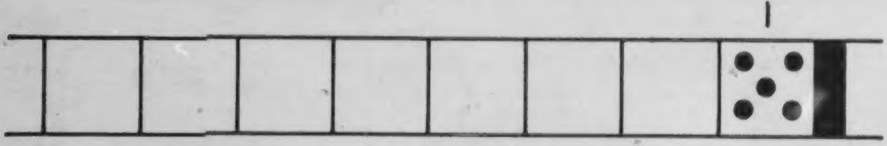
T: Let's examine a Base Five abacus.

Put one checker on the abacus.



T: This is 1.

Label the board "1" and put four more checkers on the ones' board.



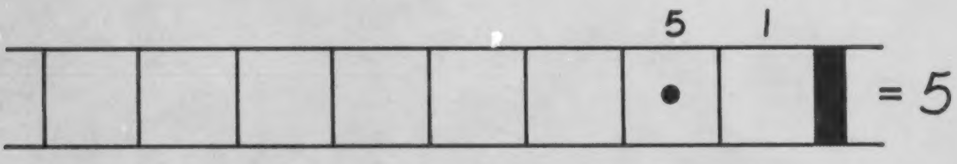
14 N

T: What number is this?

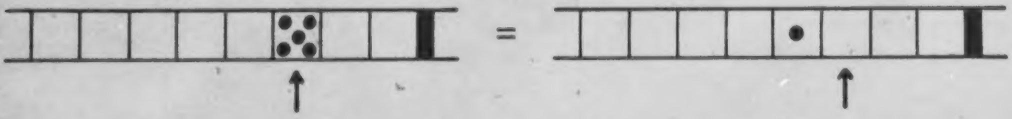
S: 5.

T: Can you put 5 on the abacus using exactly one checker?

Invite a student to make the play that puts 5 on the abacus with exactly one checker and then label the board "5".

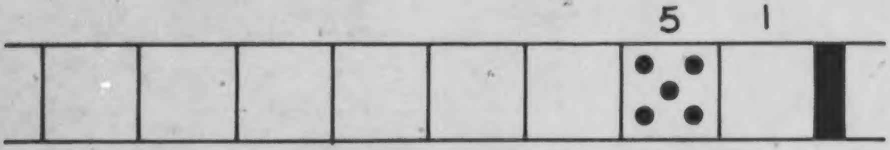


As this is being done, mention the Base Five abacus rule.



Five checkers on a board represents the same number as one checker on the next board to the left.

Put four more checkers on the fives' board.

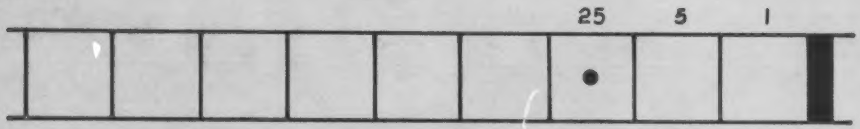


T: What number is this?

S: 25.

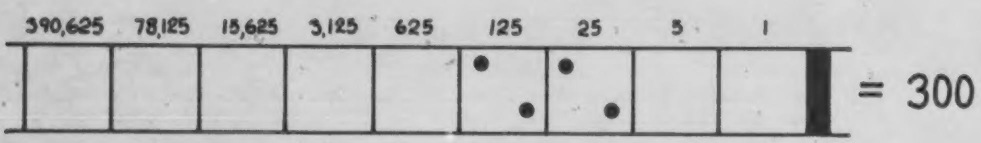
T: Can you put on 25 using exactly one checker?

Invite a student to make the play that puts on 25 with exactly one checker and label the board "25".

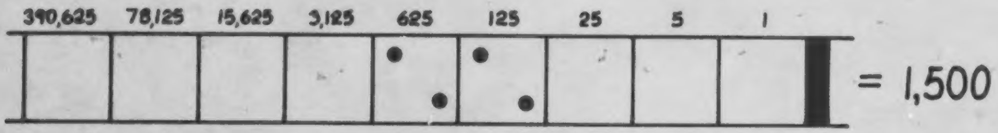


Continue in this manner until at least nine boards are labeled. Encourage the students to do the calculations of these products on their paper. Check each calculation collectively on the chalkboard.

T: Who can put 300 on the abacus using at most four checkers on any board?


390,625
78,125
15,625
3,125
625
125
25
5
1
= 300

T: Watch closely as I move each checker one board to the left. What number is this?

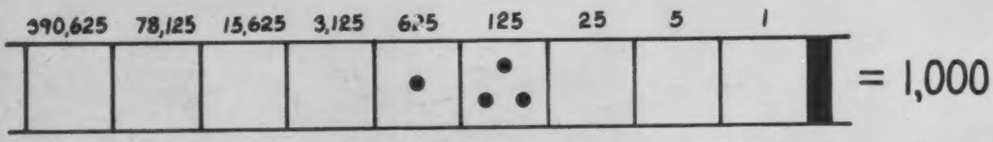

390,625
78,125
15,625
3,125
625
125
25
5
1
= 1,500

14 N

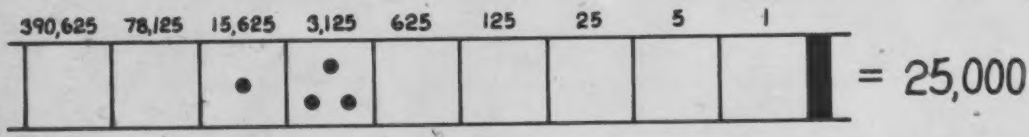
S: The number is 1,500. Moving each checker one board to the left is the same as multiplying by 5, and 5 x 300 = 1,500.

T: Draw an abacus on your paper and put 1,000 on the abacus.

Let the students have a few minutes to work and then ask someone to put 1,000 on the demonstration abacus.


390,625
78,125
15,625
3,125
625
125
25
5
1
= 1,000

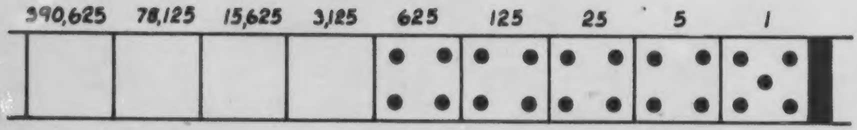
T: Good. Watch carefully as I move each checker two boards to the left. What number is this? Try to do as little calculating as possible.


390,625
78,125
15,625
3,125
625
125
25
5
1
= 25,000

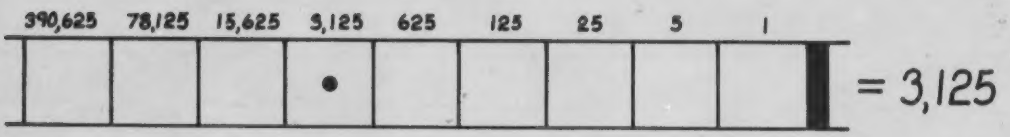
S: The number is 25,000. Moving the checkers two boards to the left is the same as multiplying by 25.

A common incorrect answer is 10,000. If many students make this error, remind them that 5 x followed by 5 x is 25 x, not 10 x.

Put these checkers on the abacus and ask the students to write the number on their paper doing as little calculating as possible.


390,625
78,125
15,625
3,125
625
125
25
5
1

Many students will be able to make these plays mentally. For the benefit of those students who have difficulty seeing the plays, ask a student to make the plays.

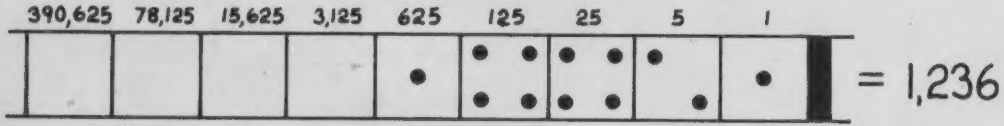

390,625
78,125
15,625
3,125
625
125
25
5
1
= 3,125

Distribute Worksheet N24.

T: Put these numbers on your abacus.

Let the students work independently for a while as you monitor their progress, offering assistance where needed. Answers for these problems are at the end of this lesson (see page 103).

As the students are finishing their work, check collectively a solution for 1,236.

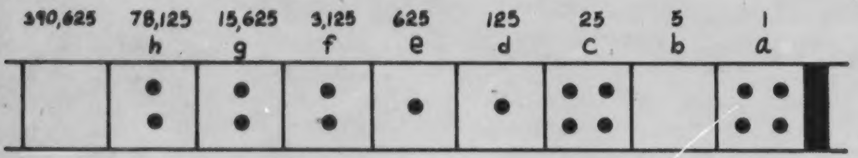

390,625
78,125
15,625
3,125
625
125
25
5
1
= 1,236

14 N

Exercise 2

T: Now that we are experts on the base five abacus, let's see how Natasha uses it to send her secret message.

Redraw this abacus from Exercise 1 and remind the class of the arrow diagram that led to this configuration.



T: What code number would Natasha send to headquarters?

S: We must add.

Write the addition problem on the board and ask a student to do the calculation.

$$\begin{array}{rrrr}
2 \times & 78{,}125 & = & 156{,}250 \\
2 \times & 15{,}625 & = & 31{,}250 \\
2 \times & 3{,}125 & = & 6{,}250 \\
 & & & 625 \\
 & & & 125 \\
4 \times & 25 & = & 100 \\
4 \times & 1 & = & 4 \\
\hline
 & & & 194{,}604
\end{array}$$

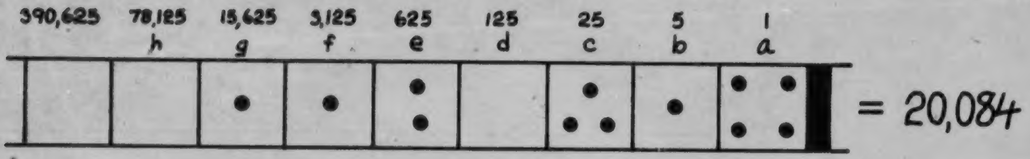
T: So Natasha would send the following message:

194,604
code 5

T: One day headquarters received this message from Natasha. Can you decode it?

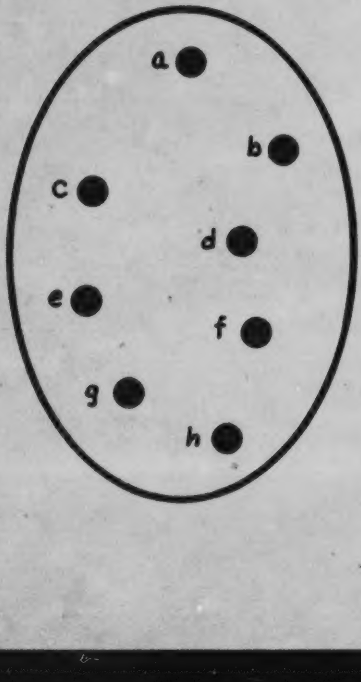
20,084
code 5

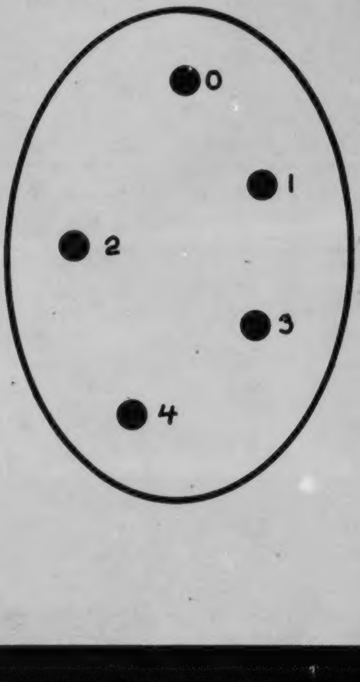
A student should suggest that this number be put on the Base Five abacus. Call on a student who has completed Worksheet N24 to put this number on the abacus.



390,625
78,125
h
15,625
g
3,125
f
625
e
125
d
25
c
5
b
1
a
= 20,084

Draw these strings and dots.

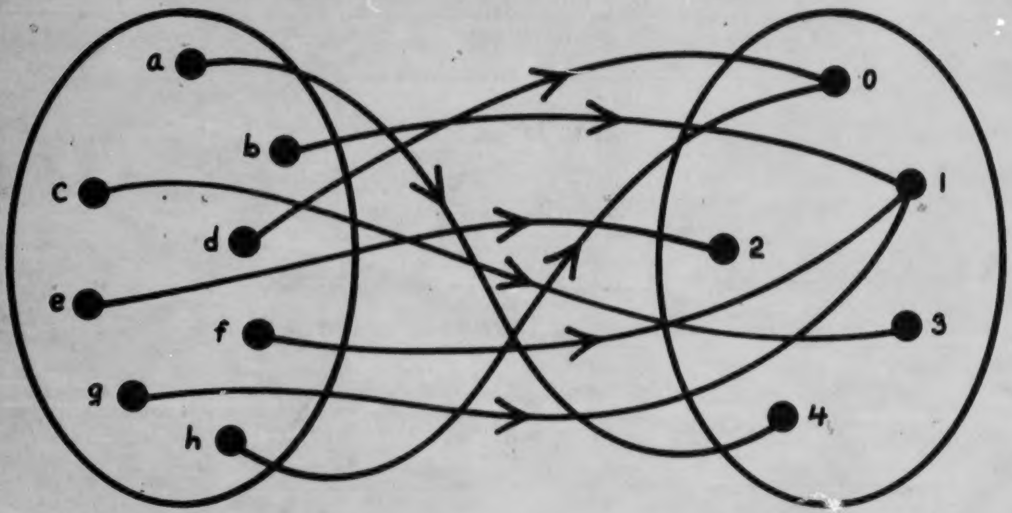




T: Which bridge did a observe?

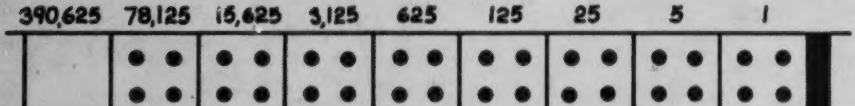
S: 4, because there are four checkers on the ones' board.

Invite students to draw the arrows until your arrow picture is complete.


a
b
c
d
e
f
g
h
0
1
2
3
4

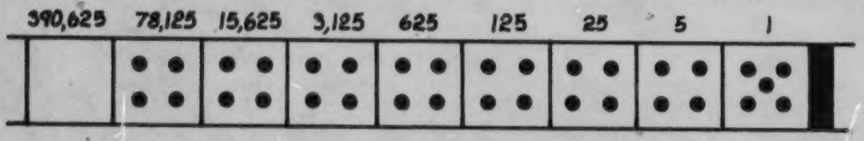
T: Which is the largest number that Natasha could send?

Ask a student to put this number on the abacus.


390,625
78,125
15,625
3,125
625
125
25
5
1

T: What number is this? Can you think of an easy way of finding out without doing many calculations?

Let the students comment on this situation. If no one suggests it, put one more checker on the ones' board.



T: This number is one more than the largest number that can be sent. What number is this?

Some students will probably suggest making the plays. They should conclude that this number is 390,625.

T: 390,625 is one more than the largest number that can be sent. So which is the largest number that can be sent?

S: 390,624.

T: Which is the smallest number?

S: 0.

T: When would 0 be sent as a message?

S: When all of the spies are observing bridge 0.

T: For each number, there is a message and for each message, there is a number. How many numbers from 0 to 390,624?

S: 390,625.

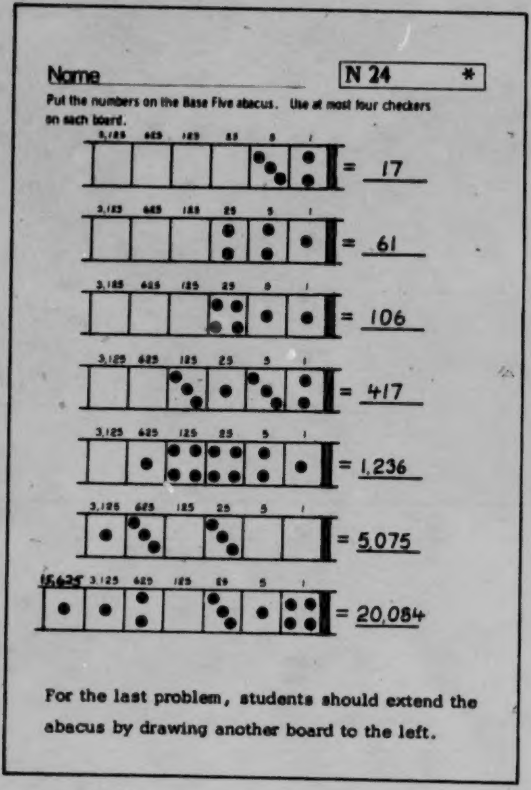
T: How many messages can Natasha send?

14 N

S: 390,625.

T: If Natasha were to send one different message each day, about how long would it take her to send 390,625 messages?

S: Over 1,000 years, because there are 365 days in a year and 1,000 x 365 = 365,000.



Name

N 24 *

Put the numbers on the Base Five abacus. Use at most four checkers on each board.

3,125 625 125 25 5 1
15,625

= 17

= 61

= 106

= 417

= 1,236

= 5,075

= 20,084

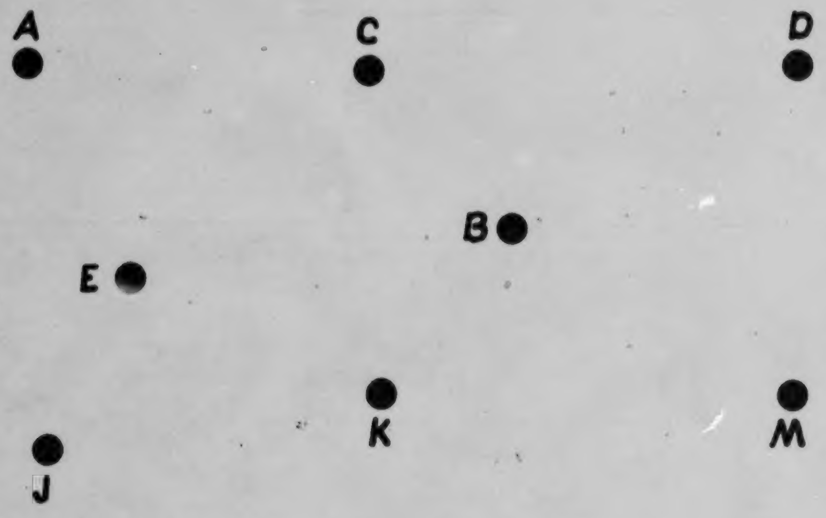
For the last problem, students should extend the abacus by drawing another board to the left.

3.4 How Many Permutations?

In preparation for this lesson, choose eight students in your class whose first names start with different letters. Write each name on a separate piece of note-size paper and fold it so that the name is not visible.

To make the description of the lesson easier to follow, the names of eight students in one of our development classes will be used: Angela, Dirk, Charles, Edward, James, Beth, Martha, and Kim.

Ask the eight students to stand at the front of the class. Draw a dot for each of them on the board. Label the dots with the first letters of the students' names.



T: I have eight pieces of paper. Each piece of paper has one of your names written on it. I will mix up the pieces of paper and give one to each of you. Do not open it yet. When I say "go", unfold the paper and read the name. Then, with your right hand take the left hand of the person whose name is on your piece of paper.

Pass out the papers and say "go!"

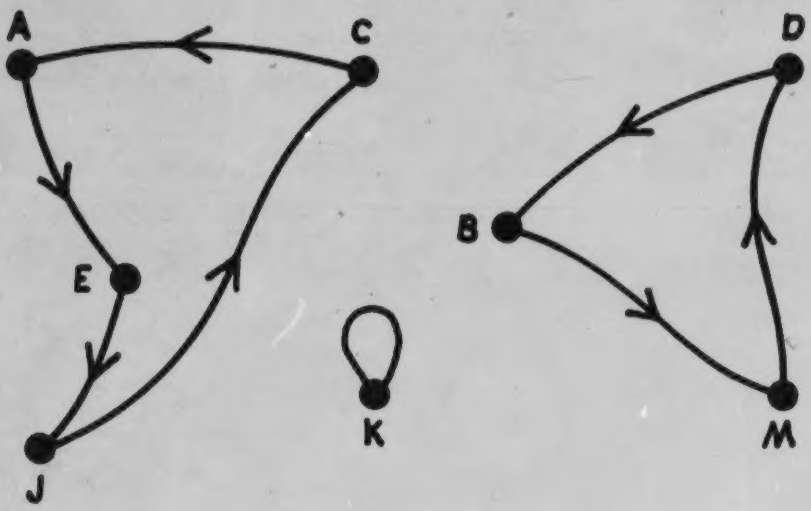
Watch the students to make sure they follow instructions.

T: I would like to draw some red arrows for "I hold your left hand with my right hand." Who can draw a red arrow in our picture on the board?

S: Charles is holding Angela's left hand. We can draw an arrow from C to A.

S: Kim received her own name -- there should be a loop at K.

Invite students not participating in the game to draw arrows and complete the arrow picture. Suppose your arrow picture looks like this.



NOTE: Each dot in your picture must have exactly one arrow starting there and exactly one arrow ending there.

Ask the eight students to take their seats.

T: Look at the arrow picture. Is it a function?

S: Yes, at each dot at most one arrow starts.

T: Good. In fact, at each dot exactly one arrow starts. What do you notice about the arrows that end at each dot?

S: There is exactly one arrow that ends at each dot.

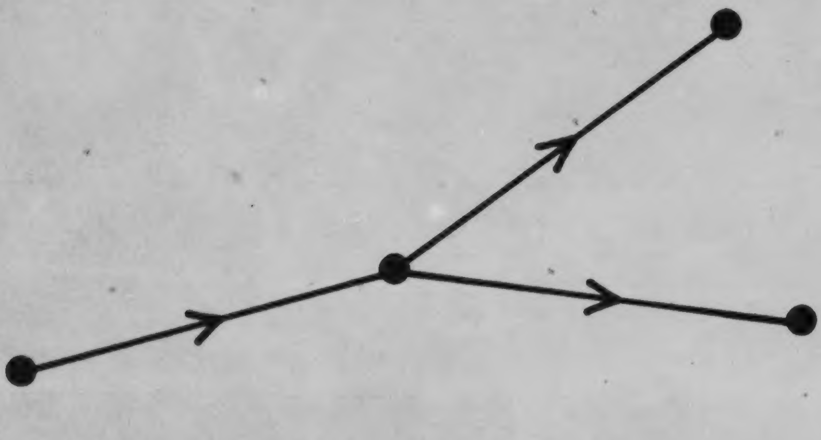
T: Each dot has exactly one arrow starting there and exactly one arrow ending there.

T: On your paper, draw eight well-spaced dots. Label them with the letters we used on the board.

Imagine that we play the game a second time. Draw an arrow picture for this imaginary game. Your picture should not be exactly the same as the one on the board. If we play the game again, it probably will have a different result from the first game.

Let students have a few minutes to work on their arrow pictures. Walk around and monitor student progress. Give help to those who have difficulty getting started.

A few students may have difficulty drawing an appropriate arrow picture. For example, students may have this arrow picture as part of their drawings.



T: What do you think of this situation?

S: That's impossible. No one has three hands.

S: The arrow picture shows someone with two right hands. That can't happen.

After many students have completed their pictures, call the class' attention back to the picture on the board.

T: Look at our arrow picture on the board. How many pieces does it have?

S: Three.

T: How many pieces do your pictures have?

S: Four pieces.

S: Seven pieces.

T: How could one have an arrow picture with seven pieces?

S: Six loops and two holding hands.

T: What is the smallest number of pieces we could have?

S: One piece. The eight students could hold hands in a circle.

T: What is the largest number of pieces?

S: Eight. Eight loops.

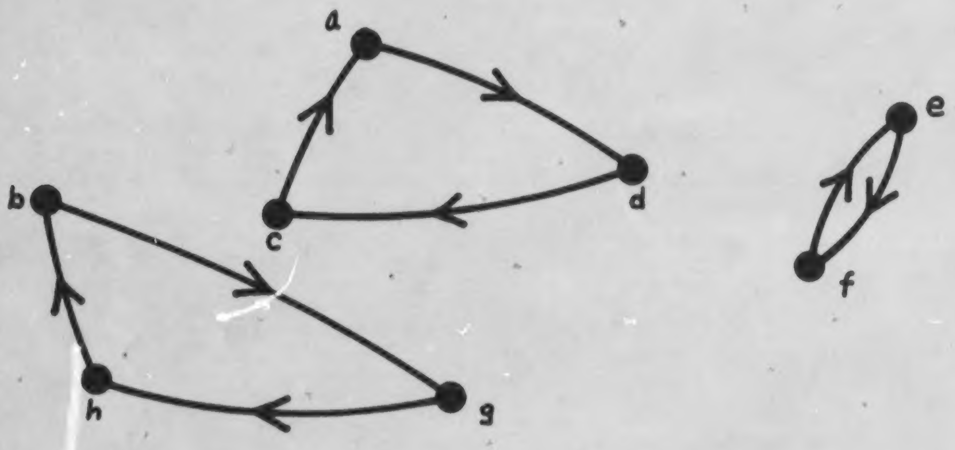
31 N

T: Can you estimate the number of different arrow pictures we could draw for eight people playing this game?

Record the students' estimates in a corner of the board. Save them for future reference.

T: Let's study this problem and try to find how many different results there could be. We will use a new way of coding.

Draw this picture on the board.



T: Suppose eight students whose first names started with a through h played our game. This arrow picture shows the outcome for one game.

Now we will code this situation.

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Put this list on the board and write the first three (corresponding) lower red letters slowly and progressively.

a	b	c	d	e	f	g	h
↓	↓	↓	↓	↓	↓	↓	↓
d	g	a					

T: What letter would go under "d"?

S: "c".

T: Why?

S: d takes c's left hand. It's like the arrow picture.

Let the students complete the correspondence to obtain the red "code word".

a	b	c	d	e	f	g	h
↓	↓	↓	↓	↓	↓	↓	↓
d	g	a	c	f	e	h	b

Highlight the code word for emphasis.

d g a c f e h b

T: This is the code word for our arrow picture.

31 N

Erase the arrows from your arrow picture and ask the students to copy the eight dots on their paper.

Write this code word on the board.

$\boxed{b \quad c \quad d \quad e \quad a \quad h \quad f \quad g}$

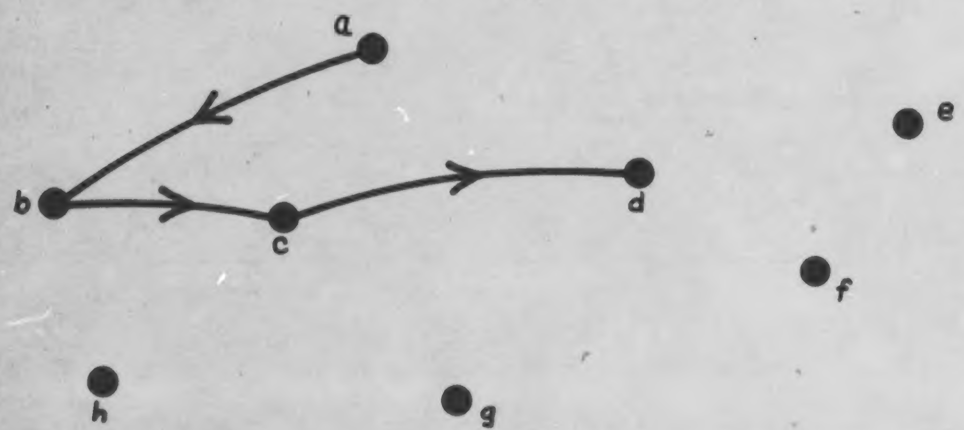
T: This is the code word of another arrow picture. Can you decode it?

Let the students have a few minutes to work on drawing the arrow picture. You could show the correspondence for the code word to help those who have difficulty.

$$\begin{array}{cccccccc} a & b & c & d & e & f & g & h \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ b & c & d & e & a & h & f & g \end{array}$$

After a few minutes, ask:

T: Can anyone draw the arrow which starts at a ? at b ? at c ?



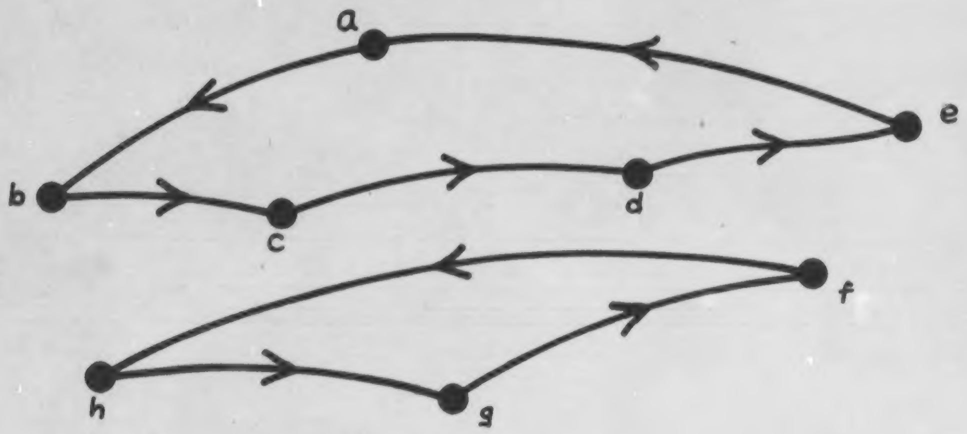
T: How does the arrow go that starts at e ?

S: From e to a.

T: How should we draw the arrow that ends at g?

S: From h to g.

Continue in this manner until the arrow picture is complete. A completed arrow picture is shown below.



Write this code word on the board.

a b c d e f g h

T: What can you tell me about the arrow picture for this code word?

S: A loop at each dot.

T: What do you notice about the code words we have written so far?

31N

S: Each of the eight letters is used exactly once in a code word.

T: Very good. For each arrow picture there is one code word, and for each code word there is one arrow picture. How many eight letter code words, using each letter exactly once, do you think there are?

Let the students have an opportunity to make an estimate or change their earlier estimate. New estimates should be recorded with the earlier estimates.

T: The problem of finding how many code words there are for eight letters is not an easy one. Let us look at a similar problem with only two letters. Write on your paper all the code words with two letters, "a" and "b", using each letter exactly once.

After a minute or two, invite a student to give the code words.

S: There are two: $\boxed{ab}$ and $\boxed{ba}$.

T: How many with three letters? Write all the code words for three letters.

After a few minutes of working independently, solve the problem collectively.

T: Which words start with "a"?

S: There are two: $\boxed{abc}$ and $\boxed{acb}$.

Write the words on the board.

$\boxed{abc}$

$\boxed{acb}$

T: Which words start with "b"?

S: bac and bca.

Add these two words to your list.

abc bac
acb bca

T: Which words start with "c"?

S: cba and cab.

Complete your list by adding these two words.

abc bac cba
acb bca cab

T: Are there any three letter code words that we have overlooked?

S: No.

T: So there are 3 × 2 or 6 code words all together.

Now try to find how many four letter code words there are. You don't have to write all of them, but try to find a way of counting them without missing any.

Encourage the students to try to find a method for counting the four letter code words. If any students finish quickly, ask them to go on finding the numbers of code words for five letters, six letters, seven letters, and eight letters.

31N

Suggest that they try and find a pattern.

Solve the four letter problem collectively.

T: How many four letter code words start with "a"?

S: Six.

You may receive many wrong answers, as well. In that case, list the six words for verification.

T: Let's list them. Give me one code word that starts with "a".

Let students give you the words as you list them on the board. The six words are:

abcd
abdc
acbd
acdb
adbc
adcb

T: How many start with "b"?

S: Six.

T: I will not list those six; just pretend they are on the board.
How many start with "c"?

S: Six.

T: And how many with "d"?

S: Six.

T: Good. How many four letter code words all together?

S: 24 or 6 x 4.

T: Very good. Now try to find the number of code words for five letters and for six letters.

While the students are working independently, put this table on the board.

Number of Letters	Number of Code Words
2	2
3	3 × 2 = 6
4	4 × 6 = 24
5	
6	
7	
8	

T: We know there are exactly 24 code words with four letters. How many code words with five letters start with "a"?

S: 24.

T: How many with "b"?

S: 24. The same for "c", "d", and "e" -- 24.

31N

T: Very good. How many five letter words all together?

S: 5 x 24 = 120.

Add this information to the table.

T: Do you notice any patterns in the table?

Number of Letters	Number of Code Words
2	2
3	3 × 2 = 6
4	4 × 6 = 24
5	5 × 24 = 120
6	
7	
8	

S: Look at the products. For four letters it's 4 x 6, for five letters it's 5 x 24, for six letters it's 6 times the product for five letters (i.e. 6 x 120).

Accept any explanation along these lines. Once a pattern is established, ask the students to complete the table. A completed table is shown below.

Number of Letters	Number of Code Words
2	2
3	3 × 2 = 6
4	4 × 6 = 24
5	5 × 24 = 120
6	6 × 120 = 720
7	7 × 720 = 5,040
8	8 × 5,040 = 40,320

T: There are 40,320 code words with eight letters. How many different arrow pictures?

S: 40,320.

Be sure to compare 40,320 to the students' estimates that are on the board.

33 N

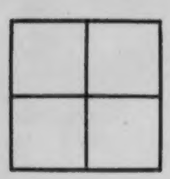
3.5 Random Art #1

NOTE: In preparation for this lesson, cut out the eight demonstration pictures with four squares that are provided on the World of Numbers poster #1 for IG-III.

Exercise 1

T: Today I would like to tell you another story about Nabu. One of Nabu's interests is painting. He does not paint portraits or landscapes; he paints random pictures of red and blue squares. Let me show you how he paints a picture.

Draw this picture.

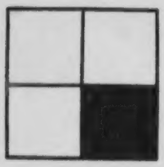


T: Nabu draws this outline of a picture. Then he takes a red marble and a blue marble in his hands and shakes them. He puts them behind his back and chooses one marble to bring forward.

Act out with marbles this part of the story as you tell it. Suppose the red marble is chosen first.

T: The red marble was selected so Nabu paints the first square (in the lower right corner) red.

Color your picture according to the color of the marble you choose first.



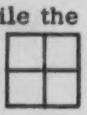
T: Nabu puts both marbles back in his hands and shakes them behind his back. He again chooses one marble to bring forward. He colors the next square with the color of the marble he chooses.

Repeat the shaking and drawing three more times and color your picture according to the results. In one of our development classes, the first picture looked like this.

T: How many different pictures of four red and blue squares could Nabu paint?

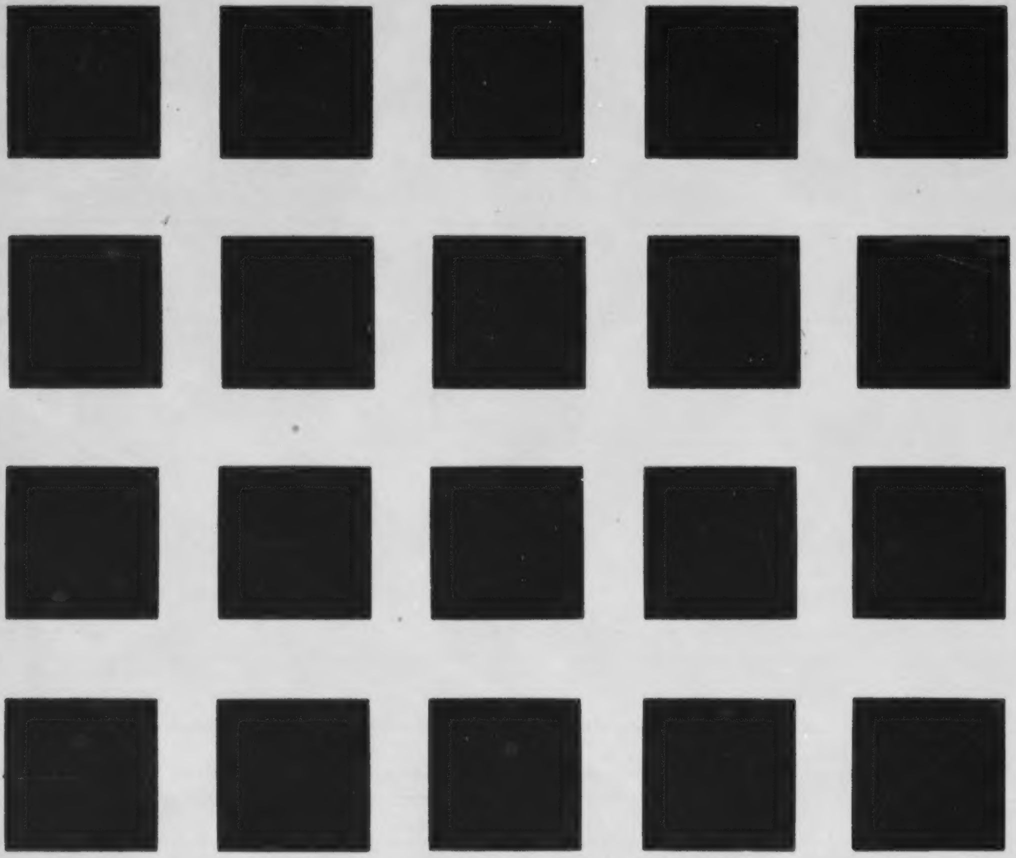
Record the estimates in one corner of your board for future reference. Distribute Worksheet N25.

T: Color as many different pictures that Nabu could paint as you can.

While the students are working independently on this problem, draw 19 more "  " on the board. One at a time, ask students to put on the board one of the colored pictures on their worksheets. Each time a student goes to the board, ask the student to try to color a picture different from the ones already on the board, but do not worry for the moment if there are some duplications.

We cannot predict the order or which duplications of pictures will be colored. To make the description of the lesson easier to follow, we suppose these were the pictures that were drawn by the students.

33 N



T: How good were your estimates of the number of different pictures?

Let the students discuss this. The students might see that there are some duplicates on the board (for example, appears twice in our display of pictures). They might decide that estimates in the 15 to 20 range are close to the actual number of pictures. In any case, lead the discussion to the suggestion that what is needed is a systematic way of checking that we have all the different pictures and no duplicates.

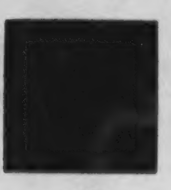
T: I have an idea of a code that will help us determine how many different pictures Nabu could paint; it will also reveal duplicates if we have any. Watch as I number the pictures and see if you can discover my code.

Slowly and progressively number the pictures according to this code:

> Red squares represent squares of the ones' board of the Minicomputer with one regular positive checker on them. Blue squares are empty squares.

For example,

11



12

1

8

6

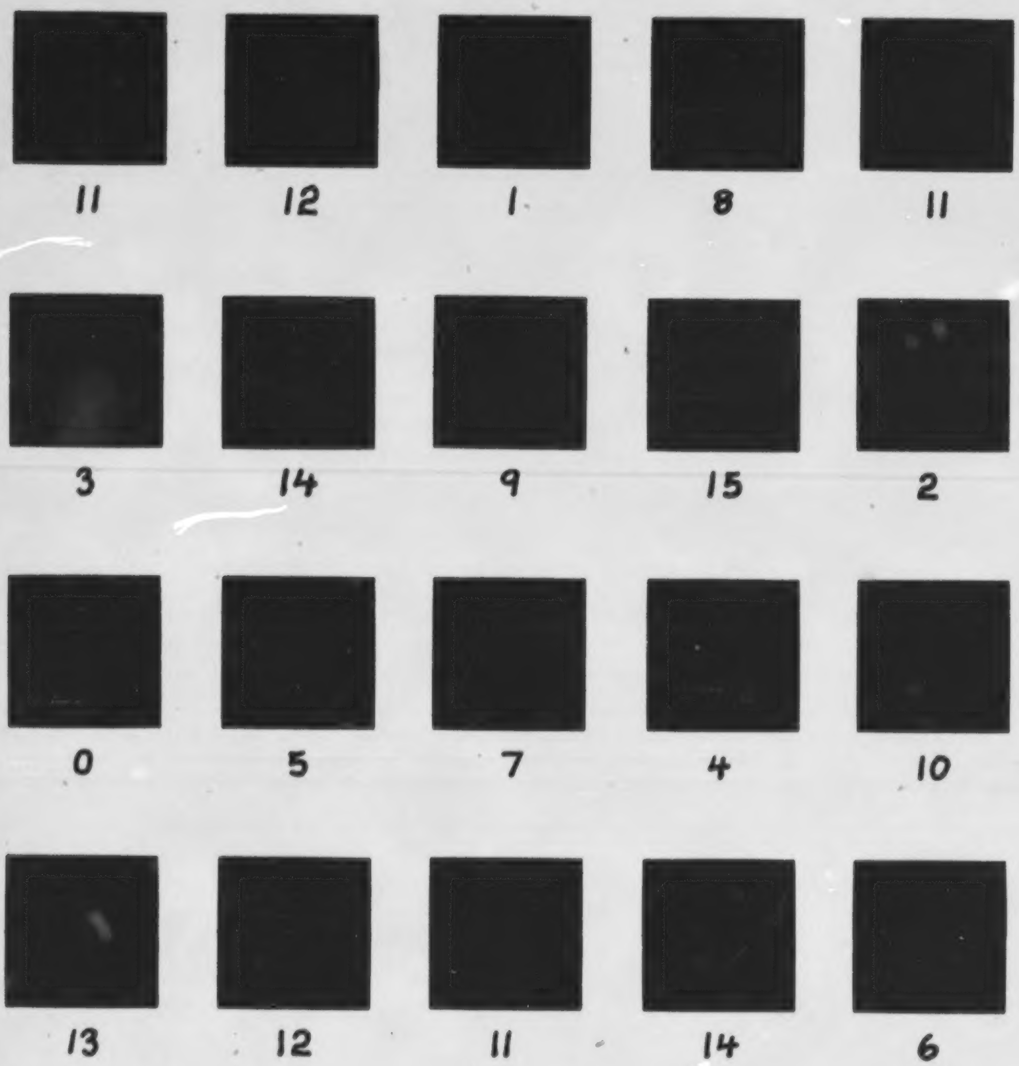
After you have numbered two or three pictures, let students try to guess what the next picture will be numbered, but do not let them give away your code. If after nine or ten pictures have been numbered, some students still have not discovered the code, give the following hint:

T: Think about the Minicomputer.

Continue numbering the pictures and letting students guess the number. When only a few pictures remain to be numbered, ask a student to explain the code. Then all the students will have a chance to try out the code on the last few pictures.

33 N

With all the pictures as on page 120 numbered, the board would look like this.


11
12
1
8
11
3
14
9
15
2
0
5
7
4
10
13
12
11
14
6

T: What is the largest number we can give to Nabu's pictures?

S: 15; the picture with all four squares painted red.

T: What is the smallest number?

S: 0; the picture with all four squares painted blue.

T: Do you think there is a picture for all the numbers between 0 and 15? (Yes)

How does this code help us to check for duplicates? (Pictures with the same number are duplicates.)

How does this code help us find how many pictures Nabu can paint?

S: There is a picture for each number from 0 to 15, and for each number from 0 to 15 there is just one picture.

T: How many different pictures are there?

S: 16, because there are 16 numbers from 0 to 15.

T: Nabu enjoys painting, but he also likes to sell his pictures. Nabu knows that he should get more money for rare pictures and less for those that are more common. What do you think a rare painting would be?

Let the class discuss what the word "rare" means and how Nabu should decide which of the pictures he could paint would be rare. Students might suggest that picture number 0 or 15 is rare because it is the smallest or largest number possible. Other students might suggest that number 8 is not rare, because it is a lot like numbers 1, 2 and 4. Accept any comments, and then procede with the following discussion which gets at the latter idea.

T: Suppose Nabu is going to paint a picture. He uses the red and blue marbles to decide how to color each square. What is the probability that Nabu will paint picture number 6?

S: $\frac{1}{16}$ (or 1 out of 16).

T: What is the probability he will paint picture number 12?

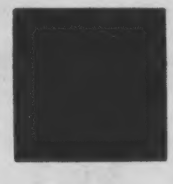
S: $\frac{1}{16}$.

33 N

T: What is the probability he will paint any one of these pictures?

S: $\frac{1}{16}$.

Display the demonstration painting number 7.



T: One day Nabu sold picture number 7 to a man who thought he was buying a one-of-a-kind painting. What do you think?

Let the students react; a student might suggest that by turning the painting, it could also be number 11 or number 13 or number 14. If this does not happen, turn the painting yourself and let the students note the different possibilities.

S: Number 7 can be turned to make number 11 or number 13 or number 14. They are really all the same painting.

List these four numbers in a box on the board.

$\boxed{7, 11, 13, 14}$

T: What is the probability that the picture Nabu paints has one of these numbers.

S: 4 out of 16 or $\frac{1}{4}$.

Write "$\frac{1}{4}$" next to the box for this list of numbers.

Display demonstration painting number 8.

T: Suppose the man had bought number 8. Do you think that it is a one-of-a-kind painting?

S: No, it is the same as number 2 and number 1 and number 4. Just turn it.

List these four numbers in another box on the board.

T: What is the probability that the picture Nabu paints has one of these four numbers.

S: $\frac{1}{4}$.

Write "$\frac{1}{4}$" next to the box for this list of numbers.

7, 11, 13, 14 $\frac{1}{4}$

1, 2, 4, 8 $\frac{1}{4}$

T: What do you think about number 0?

S: It's a one-of-a-kind.

33 N

Continue in this way, letting the students classify the pictures and giving the probabilities for each classification. Your lists on the board should be:

$\boxed{7, 11, 13, 14}$	$\frac{1}{4}$	$\boxed{0}$	$\frac{1}{16}$
$\boxed{1, 2, 4, 8}$	$\frac{1}{4}$	$\boxed{15}$	$\frac{1}{16}$
$\boxed{3, 5, 10, 12}$	$\frac{1}{4}$	$\boxed{6, 9}$	$\frac{1}{8}$

T: Suppose Nabu has painted 160 pictures using his red and blue marbles to decide the colors of the squares. About how many would be like number 7 (11, 13, and 14)?

S: 40.

Repeat this question for each classification; answers are given below in parentheses.

$\boxed{7, 11, 13, 14}$	$\frac{1}{4}$ (40)	$\boxed{0}$	$\frac{1}{16}$ (10)
$\boxed{1, 2, 4, 8}$	$\frac{1}{4}$ (40)	$\boxed{15}$	$\frac{1}{16}$ (10)
$\boxed{3, 5, 10, 12}$	$\frac{1}{4}$ (40)	$\boxed{6, 9}$	$\frac{1}{8}$ (20)

T: Which pictures are the rarest? (0 and 15)

Which are the most common? (1, 3, 7, and their likes)

T: Nabu decided to price his pictures according to how rare they are. He will charge \$20 for the rarest pictures, 0 and 15. How much should he charge for the pictures with code numbers 7, 11, 13, and 14?

S: \$5 each. There are four alike and $\frac{1}{4} \times 20 = 5$.

T: How much for 3, 5, 10, and 12 and for 1, 2, 4, and 8? (\$5 each)

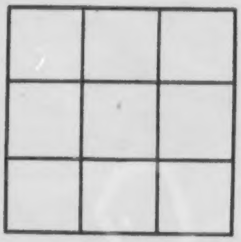
T: How much should he charge for the pictures with code numbers 6 and 9?

S: \$10 each. There are two alike and $\frac{1}{2} \times 20 = 10$.

Exercise 2

T: Nabu had soon painted and sold at least one of each kind of picture with four squares. He was getting tired of these pictures, and his customers were no longer interested in this kind of painting since they did not want to have the same paintings as other people. Nabu decided to try a new style; this time he would start with a square divided into nine smaller squares.

Draw this outline on the board.



33 N

T: Again, Nabu will use his red and blue marbles to decide how to color each square. How many different pictures of this type do you think Nabu could paint?

Record the estimates somewhere on the board for later reference.

T: Can you find a code that Nabu could use?

S: Give each square a number: 1, 2, 4, 8, 16, 32, 64, 128, 256.

This suggestion may not be forthcoming and you may have to start the process of assigning numbers to the squares. Begin in the lower right hand corner and label the squares, step by step asking the students to make suggestions as you proceed.

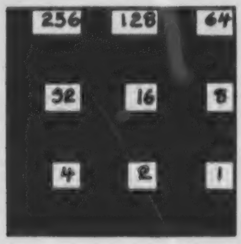
256	128	64
32	16	8
4	2	1

S: It's like the binary abacus, only in a square rather than a line.

T: Very good. How can we use these numbers to make a code.

S: A square colored red will be a square with a checker on it. A square colored blue will be an empty square.

Color your picture as shown below.

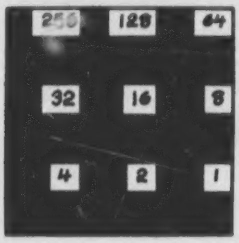


T: Which code number would this picture have?

S: 74.

T: Who can draw the picture whose code number is 22?

Invite a student to come to the board and to draw the picture shown below.



T: How many different pictures of nine red and blue squares could Nabu paint?

Let the students have an opportunity to change their estimates. You will probably find that most students will increase their estimates. Do not expect the students to find exactly that there are 512 such pictures and do not give the answer away at this time. In the next lesson, this problem will be solved completely.

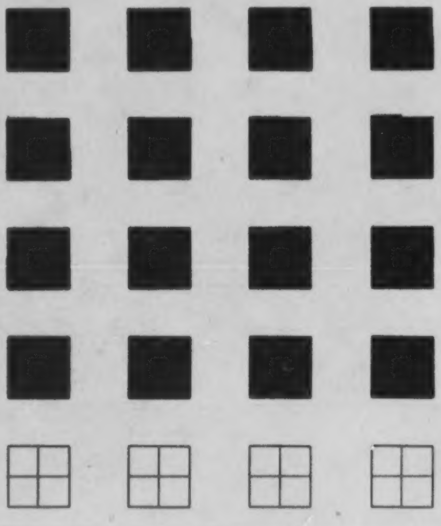
33 N

Worksheets N26 and N27 are available.

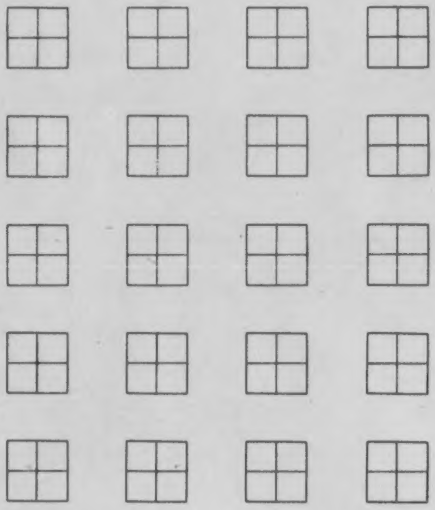
Check the worksheets collectively, if time permits. Tell the students that our story of Nabu is not complete. Nabu is interested in knowing how many pictures are rare and how many two-of-a-kind pictures there are. These problems will be investigated in a later lesson.

156

Name ______________________ N 25 *

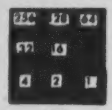


(There are 16 different pictures.)



Name ______________________ N 26 **

Label each picture with its code number.



17

84

257

0

170

325

Name ______________________ N 27 ***

Color the pictures which have these code numbers.

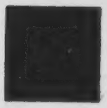
12

50

20

100

75



300

3.6 Random Art #2

NOTE: In preparation for this lesson, cut out the twelve demonstration pictures with nine squares that are provided on the World of Numbers poster #1 for IG-III.

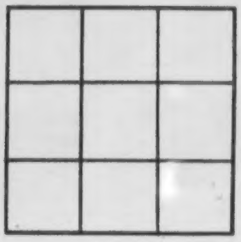
Exercise 1

T: Do you remember the story of Nabu and his paintings?

Let the students tell as much of the story of the previous lesson as they recall. They should be reminded that for the pictures of four red and blue squares, there are 16 ways for Nabu to paint a picture; but if you consider that some pictures are the same as other pictures when they are turned, there are only six different pictures.

T: After Nabu got tired of painting pictures of four red and blue squares, he decided to create some pictures of nine red and blue squares in exactly the same way.

Draw this outline on the board.



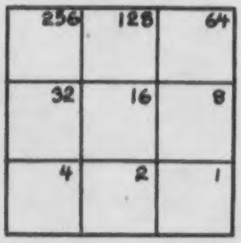
T: How many ways do you think Nabu could paint a picture?

Record some estimates in a corner of the board for future reference.

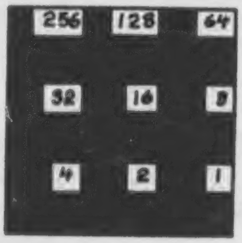
T: Do you remember the code we invented to number the pictures Nabu could paint?

S: The squares were numbered like a binary abacus: 1, 2, 4, 8, 16, 32, 64, 128, 256.

Label the squares of your picture.



Color your picture as shown below.



T: What is the code number of this picture?

S: 152, because 128 + 16 + 8 = 152.

T: I'd like someone to put the picture whose code number is 175 on the board.

35 N

Invite a student to come to the board and to draw the picture shown below.

175

T: Which is the largest code number one of Nabu's paintings could have?

Let the students have a few minutes to consider this question.

S: 511.

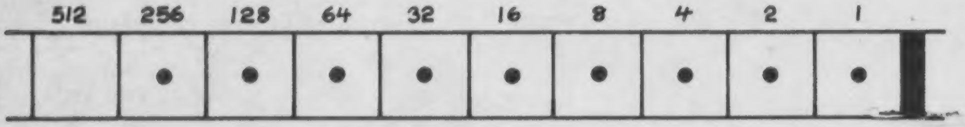
S: 512.

T: Someone come to the board and paint the picture with the largest code number.

S: Color all the squares red; it's the same as putting one checker on each square of a binary abacus.

T: That's a good idea. What number is this?

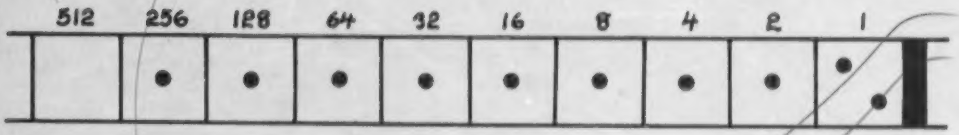
Draw a binary abacus (or use Minicomputer boards) and put on this configuration.



512	256	128	64	32	16	8	4	2	1
	●	●	●	●	●	●	●	●	●

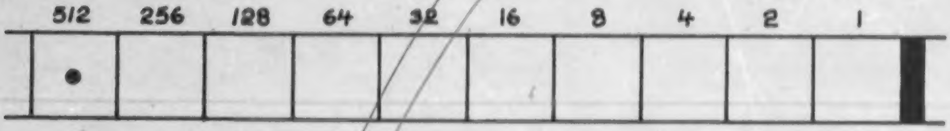
T: How can we calculate this number quickly?

S: Put another checker on the ones' board.



S: Then make plays.

Invite a student to make the plays at the board to obtain 512.



T: Is 512 the largest number?

S: No. We added an extra checker, so 512 - 1 = 511 is the largest number.

T: Very good. Which is the smallest number?

S: 0.

T: Could Nabu paint a picture for every number from 0 to 511?

S: Yes.

If your class is not convinced of this, start with 0 on the binary abacus and keep adding 1 to get successively 1, 2, 3, 4, 5, 6,.... . You will not need to continue this very long before most students realize every number between 0 and 511 is the code number for a picture.

35 N

T: How many pictures in all?

S: From 0 to 511 there are 512 numbers. So 512 pictures.

T: Nabu's customers are still interested in purchasing the rarest pictures. They would like to have one-of-a-kind pictures. What would be a one-of-a-kind picture?

S: No matter how you turn it, it is the same.

T: Can you draw some of these one-of-a-kind pictures?

Let the students work on this problem independently. As the one-of-a-kind pictures are discovered, invite students to draw them on the board. The eight such pictures are shown below and on the next page; the corresponding code number of each picture is also given. It will be easier for students to refer to pictures by number and to see that there are no duplicates.

511

0

495

16

341

170

325

186

T: We have found eight one-of-a-kind pictures. What do you notice about these pictures?

S: The pictures come in pairs. If you find one picture, then reverse the colors, red and blue, for each square and you will get another one-of-a-kind picture, e.g., 495 and 16.

A student might suggest (however, you should not force this observation) that the sum of the numbers of a "complementary" pair of pictures is 511, e.g., 495 + 16 = 511.

T: Good. Try to remember that pictures come in pairs. It will be useful to us later.

Nabu charges $60 for a one-of-a-kind picture. How much would he charge for a two-of-a-kind picture?

35 N

S: $30. $\frac{1}{2} \times 60 = 30$.

T: Let's try to find all of the two-of-a-kind pictures. Are there any two-of-a-kind pictures with exactly one red square?

S: No. Those pictures with one red square are either one-of-a-kind or four-of-a-kind.

Display the two cut-outs shown below to demonstrate the above comment.

ONE-OF-A-KIND

FOUR-OF-A-KIND

T: Are there any two-of-a-kind pictures with exactly two red squares?

Invite students to draw them on the board (there are four). To save time, suggest to the students that only the color red be used in the drawings on the board (the answers shown below will continue to use red and blue for the squares).

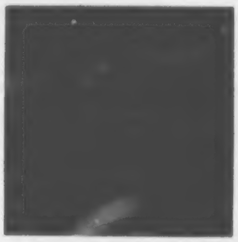
S: If you reverse the colors, red and blue, you will have four more pictures.

Ask students to draw these four complementary pictures on the board.

Continue in this manner for the two-of-a-kind pictures with exactly three red squares and their complements . . .

35 N

. . .and the two-of-a-kind pictures with exactly four red squares and their complements.



It is not necessary to consider two-of-a-kind pictures with exactly five (or more) red squares. Two-of-a-kind pictures with exactly five red squares are the complements of two-of-a-kind pictures with exactly four red squares; likewise for six red squares and three red squares; for seven red squares and two red squares and so on.

Faster students may wish to consider three-of-a-kind and four-of-a-kind pictures. They should discover that there are no three-of-a-kind pictures. Since there are 512 pictures all together, 8 one-of-a-kind pictures, and 24 two-of-a-kind pictures, the students should conclude there are 480 four-of-a-kind pictures (512 - (8 + 24) = 480).

4

MINICOMPUTER

Some "new" checkers on the Minicomputer, i.e., weighted checkers, were introduced in the CSMP Mathematics for the Intermediate Grades, Part II. These checkers extend the role of the Minicomputer in mental arithmetic to include basic multiplication facts and the types of calculations which lead to forms of the standard multiplication algorithm.

In this chapter the new checkers are first reviewed in a series of activities where numbers are put on the Minicomputer with regular, negative, and/or weighted checkers. Further experiences with the weighted checkers, in moving them to make a specified change in the number on the Minicomputer, lead to an extension of the Minicomputer Golf game.

CAPSULE LESSON SUMMARIES

Section 4.1:	Put regular and negative checkers on the Minicomputer and decode the number. Put a specified number on the Minicomputer using exactly one new checker; exactly two new checkers; one negative and one new checker. Estimate the number on the Minicomputer. Change the number by a specified amount larger or smaller by moving exactly one new checker. (Worksheets N28, N29, N30)
Section 4.2:	From a list of numbers, select those that can be put on the Minicomputer with exactly one new checker; exactly two new checkers; one negative and one new checker. Decode numbers on the Minicomputer, and by moving single checkers simulate a given arrow picture. (Worksheets N31, N32, N33, N34)
Section 4.3:	Review the process of moving a new checker to change the number on the Minicomputer by a specified amount. Play a version of the Minicomputer Golf game with new and regular checkers.

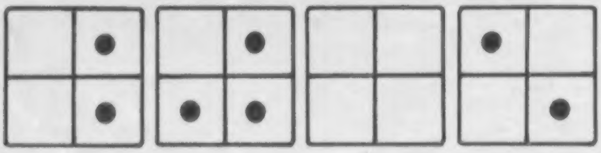
4 N

4.1 Minicomputer Golf Preparation #1

Exercise 1

If you prefer, the decoding of numbers on the Minicomputer can be performed with hand-calculators rather than with paper and pencil.

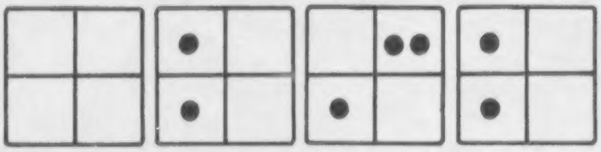
Display four Minicomputer boards and slowly put these checkers on the boards.



T: Write this number on your paper.

S: 5,709.

Put this number on the Minicomputer and have the students write it on their papers.



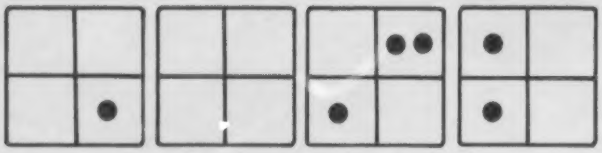
= 1,110

For those who have difficulty decoding try asking:

T: Can you show the same number using one less checker?

Have a student make a move. For example,

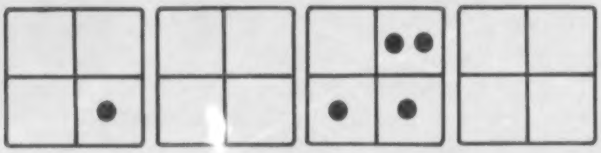
S: 800 + 200 = 1,000.



T: Can you show the same number using one less checker?

Have another student make a move. For example,

S: 8 + 2 = 10.



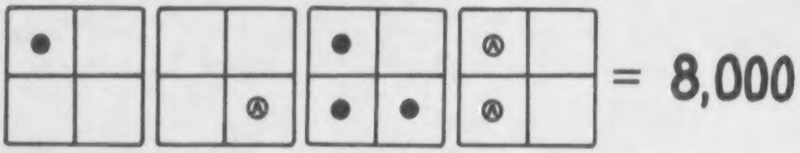
After each move, ask:

T: What number is on the Minicomputer?

S: 1,110.

If necessary, continue this line of questioning until you have the standard configuration for 1,110.

Put another number on the Minicomputer and ask the students to write it on their papers.



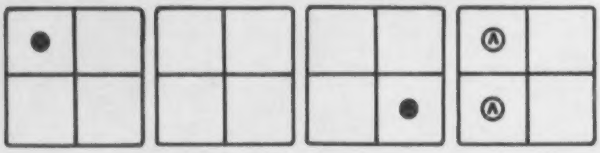
4 N

For those who have difficulty decoding, ask:

T: Can you show the same number using three less checkers?

Have a student make the moves. For example,

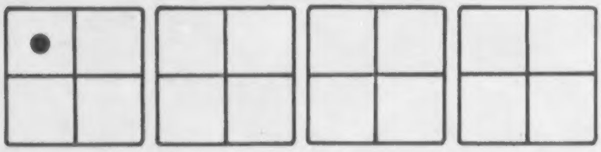
S: 80 + 20 = 100 and 100 + ⁻100 = 0.



T: Can you show the same number using three l[illegible] checkers?

Have a student make the moves.

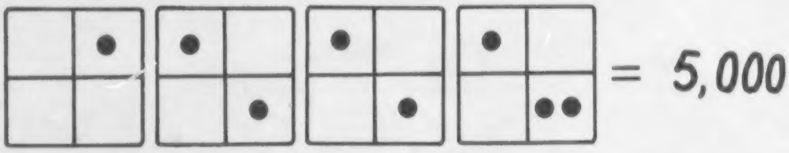
S: ⁻8 + ⁻2 = ⁻10 and ⁻10 + 10 = 0.



T: What number is on the Minicomputer?

S: 8,000.

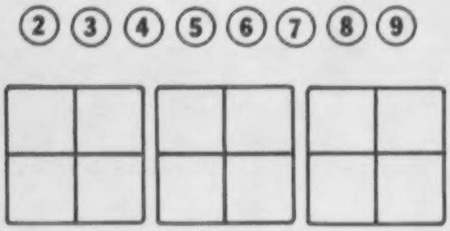
Put another number on the Minicomputer and ask the students to write it on their papers.



Use a questioning method similar to the previous one for those students who have difficulty decoding.

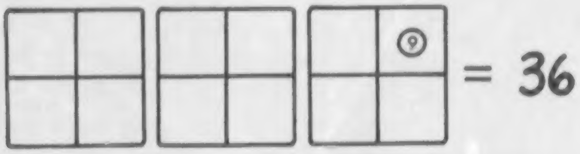
Exercise 2

Display three Minicomputer boards and your set of these checkers:

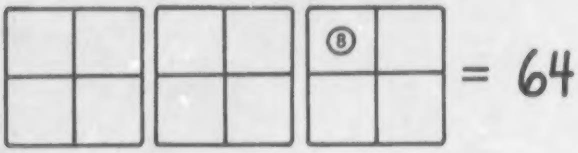


T: Can you put 36 on the Minicomputer using exactly one of the checkers ② through ⑨ ?

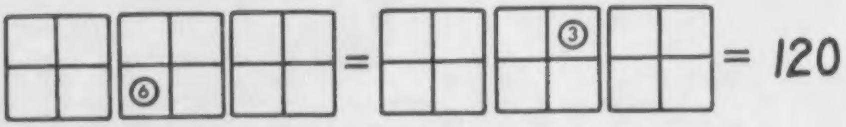
Have a student do it.



T: Put 64 on the Minicomputer using exactly one of these new checkers.



T: Put 120 on the Minicomputer using exactly one of these new checkers.

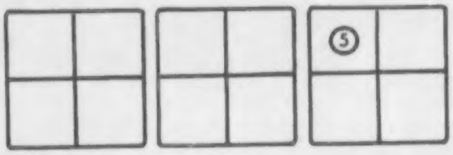


Exercise 3

Add a negative checker to the set of displayed checkers.

T: Put 30 on the Minicomputer using exactly one negative checker and one of the new checkers.

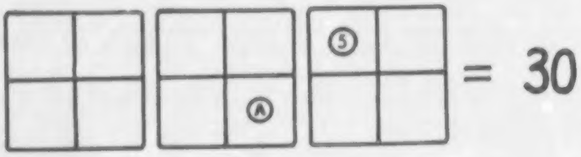
Have one student place just one of the checkers. Suppose a ⑤-checker is placed first.



T: What number is on the Minicomputer?

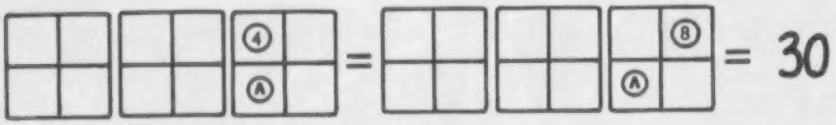
S: 40.

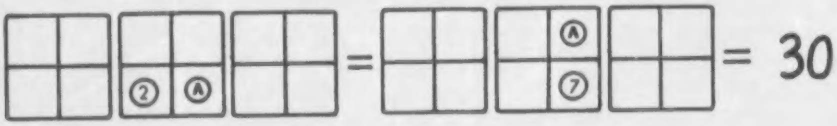
T: Who can place the negative checker?



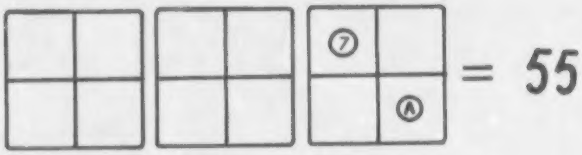
T: Can you show 30 another way with exactly one negative checker and one of the new checkers?

Have one student place the first checker and another student place the second checker. If the students offer, have them show several other ways. Below are some other ways of showing 30 with exactly one negative checker and one of the new checkers.





Repeat this exercise with 55.



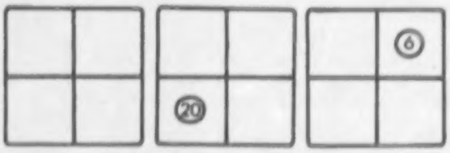
171

4-N

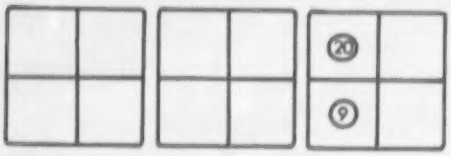
Exercise 4

Add a ⑳-checker to the set of checkers that you have displayed. Ask the students to put the following numbers on the Minicomputer using exactly one ⑳-checker and one of the new checkers - ② through ⑨. One possible answer for each problem is shown below. This exercise should be conducted in the same spirit as the previous two exercises.

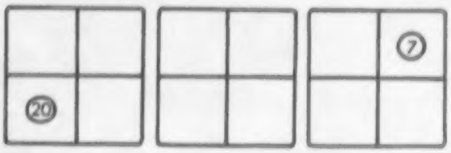
424 =



178 =

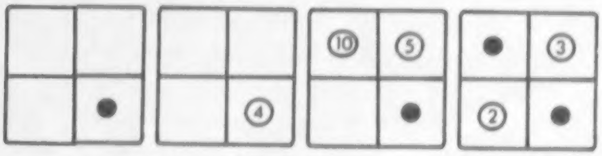


4,028 =



Exercise 5

Display four Minicomputer boards with this configuration of checkers.



175

T: Look carefully at the number I have put on the Minicomputer. What is your estimate of this number? Is it larger than 1,000?

S: Yes. There is 1,000 on the thousands' board, and there are more checkers.

T: Is it larger than 2,000?

S: Yes. A ⑩-checker on the 80-square is 800 and a ④-checker on the 100-square is 400. 800 + 400 = 1,200 so the number is at least 2,200.

T: Very good. Who can make this number 12 bigger by moving exactly one checker?

S: Move the checker from the 8-square to the 20-square.

T: Who could move a different checker and make this number 12 bigger?

S: Move the ③-checker from the 4-square to the 8-square.

Have a student make this move.

T: Do you understand this move? A ③-checker on the 4-square is how much?

S: 12.

T: I move the ③-checker to the 8-square. A ③-checker on the 8-square is how much?

S: 24.

T: How much bigger is 24 than 12?

S: 12 bigger.

T: Now I have a new number. Who can make this number 9 smaller?

Have students whisper to you the move that should be made (move the regular checker from the 10-square to the 1-square). Then have a student make the move.

T: Can you make the number 36 bigger by moving exactly one checker?

Have students whisper their move to you. (Move the ②-checker from the 2-square to the 20-square or the ③-checker from the 8-square to the 20-square.)

S: Moving a regular checker from the 2-square to the 20-square makes the number 18 bigger. Moving the ②-checker from the 2-square to the 20-square makes the number 36 bigger.

T: Very good. Can you make the number 200 bigger?

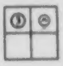
S: Move the ⑤-checker from the 40-square to the 80-square.

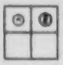
Worksheets N28, N29, and N30 are available.

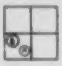
Name ______________________ **N 28** *

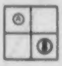
Put any number you wish on the Minicomputer using exactly one negative checker and exactly one of these checkers:

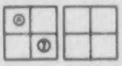
② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

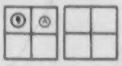

= 20

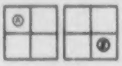

= 24

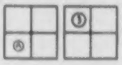

= 10

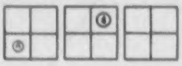

= 0


= −10


= 680


= −78


= 4

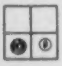

= −40

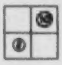
(Many solutions are possible.)

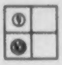
Name ______________________ **N 29** **

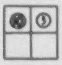
Put each number on the Minicomputer using one ⑩ - checker and exactly one of these checkers:

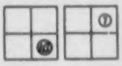
② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

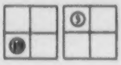

= 28

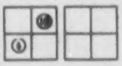

= 56

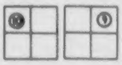

= 92

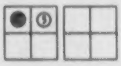

= 100

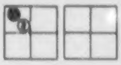

= 128


= 240


= 520


= 836

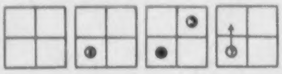

= 1,000


= 960

(Many solutions are possible.)

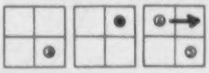
Name ______________________ **N 30** ***

By moving the ⑦ - checker from the 2 - square to the 8 - square, you make this number 42 larger.

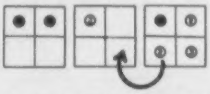


By moving exactly one checker, make the number

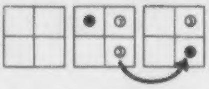
24 smaller.



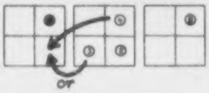
36 larger.



27 smaller.



240 larger



(There are two possible answers for the last problem.)

17 N

4.2 Minicomputer Golf Preparation #2

Exercise 1

Display two Minicomputer boards and these checkers:

●; ② through ⑨; ⑩; ⑳; ㊿; (∧)

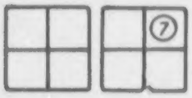
Write this list of numbers on the chalkboard.

28; 34; 36; 42; 48; 54; 56; 120; 400; 1,000

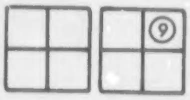
T: Which of these numbers can you put on the Minicomputer using exactly one checker?

Have students put numbers on the Minicomputer, while the rest of the class checks to see if the number on the Minicomputer is one of the numbers in the list.

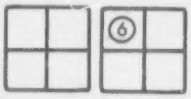
These seven numbers—28; 36; 48; 56; 120; 400; and 1,000—can be put on the Minicomputer at least one way using exactly one of the checkers that have been displayed. Below we show one way for each number.



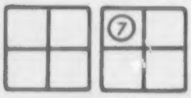
= 28



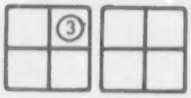
= 36



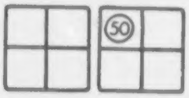
= 48



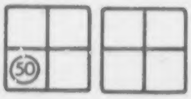
= 56



= 120



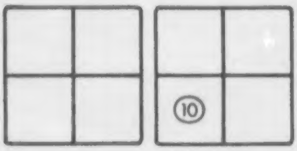
= 400



= 1,000

T: Can you put 34 on the Minicomputer using a ⑩-checker and one other checker?

Ask a student to place just one checker first. Suppose it is a ⑩-checker.



T: What number is on the Minicomputer?

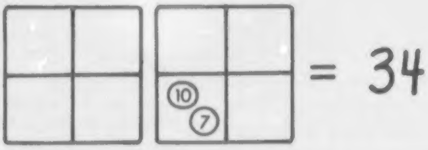
S: 20, because 10 × 2 = 20.

T: How much larger is 34 than 20?

S: 14 larger.

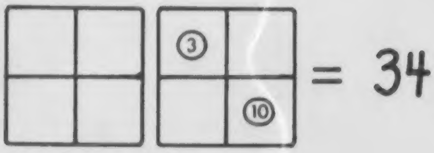
T: Can someone put 14 on the Minicomputer using exactly one of the other checkers?

Have a student place the second checker.



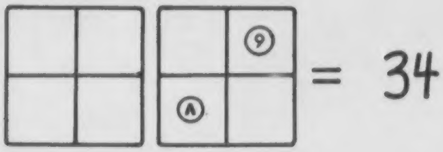
17 N

Another solution to this problem is:

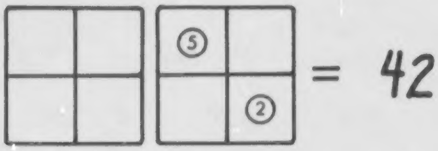


Continue in a similar manner with these problems.

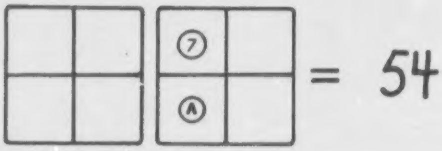
Put 34 on the Minicomputer using exactly one negative checker and one other checker.



Put 42 on the Minicomputer using a ②-checker and a ⑤-checker.

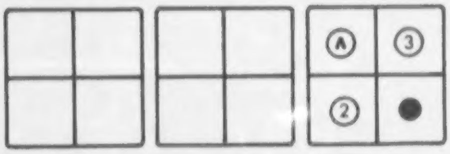


Put 54 on the Minicomputer using exactly one negative checker and one other checker.



Exercise 2

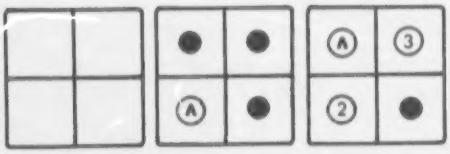
Display three Minicomputer boards with this configuration of checkers.



T: Write this number on your paper.

After a few minutes, the students should conclude that the number is 9.

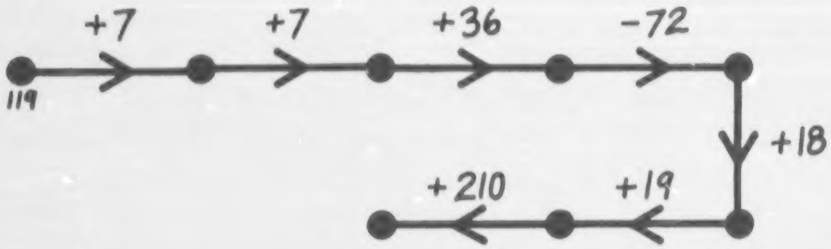
Put these additional checkers on the tens' board.



T: Write this number on your paper.

S: The number is 119; 110 + 9 = 119.

Draw this arrow picture on the board.



T: The first arrow starts from the number on the Minicomputer (119) and asks you to add 7 to the number by moving exactly one checker from one square to another square. How can you do it?

S: Move a regular positive checker from the 1-square to the 8-square.

S: Or move a negative checker from the 8-square to the 1-square.

Be sure that both moves are mentioned and make one of the moves. At this point, it makes no difference which move is made — later it will make a difference.

T: What number is on the Minicomputer?

S: 126.

Label the second dot "126".

T: What do we have to do next?

S: Move one checker to make the number 7 greater.

T: How can that be done?

Probably the students will suggest making the move that was not made for the first + 7 arrow. Have a student make the move.

T: What number is on the Minicomputer now?

S: 126 + 7 = 133.

183

Label the next dot "133". Follow similar steps for the other arrows.

+ 36 Move the ②-checker from the 2-square to the 20-square.
The number now on the Minicomputer is 169.

− 72 Move a regular positive checker from the 80-square to the 8-square.
The number now on the Minicomputer is 97.

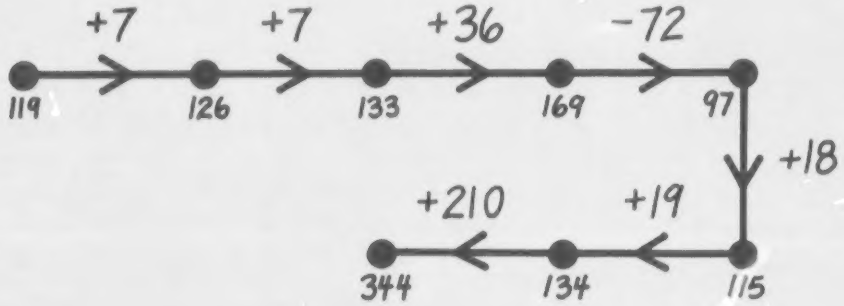
+ 18 Move a negative checker from the 20-square to the 2-square, or
move the ③-checker from the 4-square to the 10-square.
The number now on the Minicomputer is 115.

NOTE: Be sure both moves are mentioned but make the move with the ③-checker, otherwise there will be no move for the last arrow in the road.

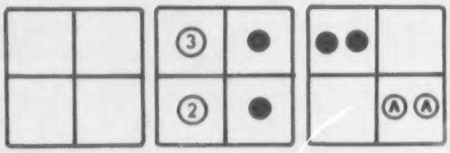
+ 19 Move a negative checker from the 20-square to the 1-square.
The number now on the Minicomputer is 134.

+ 210 Move a ③-checker from the 10-square to the 80-square.
The number now on the Minicomputer is 344.

The arrow picture has been progressively completed,



and the checkers on the Minicomputer have been moved to obtain the following configuration.



T: What number should be on the Minicomputer now?

S: 344.

Have the students calculate the number on the Minicomputer to see if it is correct.

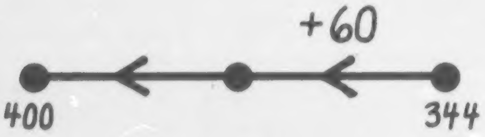
Challenge the class to start from 344 and to get 400 with the smallest number of moves.

T: What should you add to 344 to get to 400?

S: 56.

Have the students work independently on this problem for a few minutes.

If some students have difficulty with this problem, give a hint. For example, extend your arrow picture with these two arrows.



T: One way to solve this problem is to make two moves. The first move is to make the number on the Minicomputer 60 greater. How can that be done?

S: Move a ③-checker from the 80-square to the 100-square (or move a regular positive checker from the 40-square to the 100-square).

Have a student make one of these moves.

T: What number should be on the Minicomputer now?

S: 404.

Label the dot (at the end of the + 60 arrow) "404".

T: What could this last red arrow be for?

S: - 4.

Label the last arrow "- 4".

T: What move could we make?

S: Move a regular positive checker from the 8-square to the 4-square.

Make this final move and have the students calculate the number on the Minicomputer to see if it is correct.

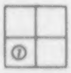
Worksheets N31, N32, N33, and N34 are available.

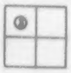
Name ______________________ **N 31** *

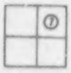
14; 19; 24; 28; 38; 40; 48; 54; 64

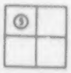
Six of these numbers can be put on the ones' board of the Minicomputer using exactly one of these checkers. Put them on.

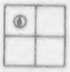
② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

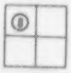

= 14


= 24


= 28


= 40


= 48

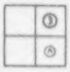

= 64

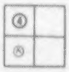
Name ______________________ **N 32** **

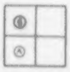
11; 21; 30; 38; 42; 47; 54; 112; 115

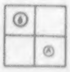
Six of these numbers can be put on these Minicomputer boards using exactly one negative checker and exactly one of these checkers. Put them on.

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨


= 11


= 30


= 38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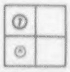
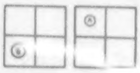
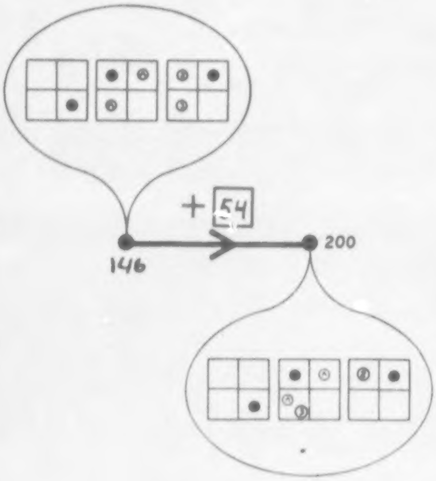

= 54


= 112

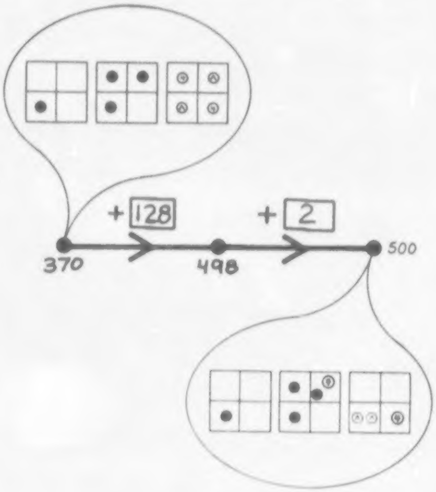
Name ______________________ **N 33** ***

Solve this puzzle by moving exactly one checker.



Name ______________________ **N 34** ****

Solve this puzzle by moving exactly two checkers, one for EACH arrow.

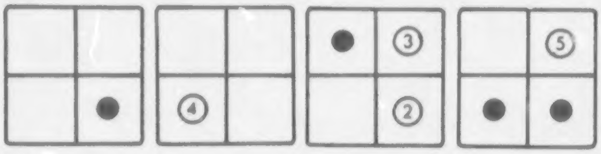


4.3 Minicomputer Golf

NOTE: This lesson may be varied and used as an adjustment day activity if you so desire.

Exercise 1

Display four Minicomputer boards with this configuration of checkers.



T: Look carefully at the number I have put on the Minicomputer. What is your estimate of this number? Is it larger than 1,500?

S: Yes, there is 1,000 on the thousands' board and 800 on the hundreds' board.

T: Very good. I will do something to the number by moving some checkers. Each time you are to tell me if I get a larger number, a smaller number, or the same number.

Move the ⑤-checker from the 4-square to the 8-square.

S: The number is larger.

T: How much larger?

S: 20 larger.

T: How did you get 20?

S: A ⑤-checker on the 4-square is 20 and a ⑤-checker on the 8-square is 40. 40 - 20 = 20, so it is 20 larger.

Continue in this manner making the following moves:

Move a ③-checker from the 40-square to the 10-square.
(90 smaller; 120 - 30 = 90)

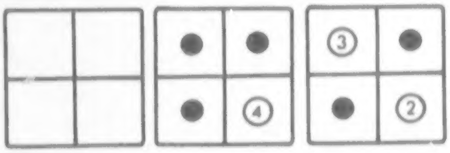
Move the regular checker from the 80-square to the 800-square.
(720 larger; 800 - 80 = 720)

Move a ⑤-checker from the 8-square to the 20-square.
(60 larger; 100 - 40 = 60)

Move the ④-checker from the 200-square to the 2-square.
(792 smaller; 800 - 8 = 792)

Exercise 2

Slowly and progressively put this number on the Minicomputer.



T: What number is this?

S: 212.

T: Now we will play a game of Minicomputer Golf and use this number as a starting number.

25 N

Divide your class into two groups. One group will be Team A and the other group will be Team B.

Write this information on the board.

Goal: 1,000

Team A →

Team B →

T: We are starting from 212 and the goal is 1,000. Students from Team A and from Team B will play in turn. Each time, they may move one checker from any square to any other square. The number will be increased or decreased. The first team to reach the goal will be the winner. Who would like to play first for Team A?

Suppose the first player for Team A moves a regular positive checker from the 40-square to the 100-square. (We reproduce below the results of a game played in one of our development classes.)

T: Is the number on the Minicomputer larger or smaller?

S: Larger.

T: How much larger?

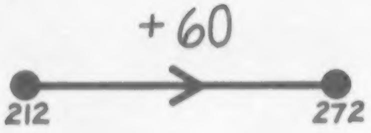
S: 60 larger.

T: What number is on the Minicomputer now?

S: 212 + 60 = 272.

25 N

Draw this red arrow on the board.



Now a player from Team B takes a turn. Suppose a ②-checker is moved from the 1-square to the 20-square.

T: Is the number larger or smaller?

S: Larger.

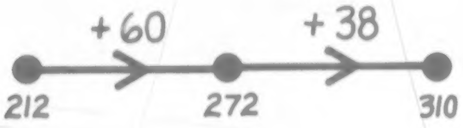
T: How much larger?

S: From 2 to 40 — it is 38 larger.

T: What is the new number?

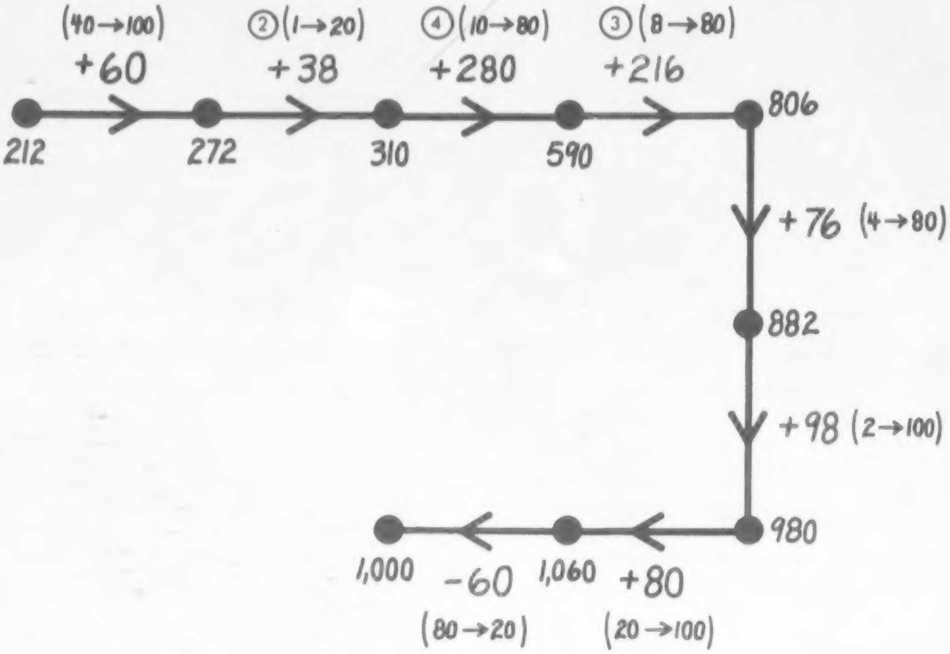
S: 272 + 38 = 310.

Add this blue arrow to the picture.



Follow similar steps for the next players, alternately from Team A and Team B. The results of our game are described by this red-blue road. Team B was the winner.

NOTE: In this picture "② (1→20)" indicates a ②-checker was moved from the 1-square to the 20-square, while "(40→100)" indicates a regular positive checker was moved from the 40-square to the 100-square.


(40→100)
+60
② (1→20)
+38
④ (10→80)
+280
③ (8→80)
+216
212
272
310
590
806
+76 (4→80)
882
+98 (2→100)
980
+80
(20→100)
1,060
-60
(80→20)
1,000

Exercise 3

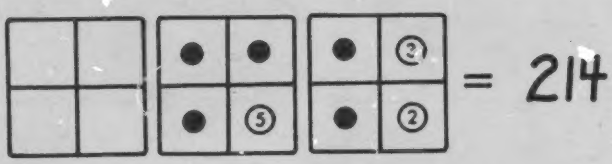
NOTE: For this activity, each pair of students will need one student Minicomputer set and one hand-calculator, if available. These materials are not supplied with the classroom set of materials for CSMP Mathematics for the Intermediate Grades, Part III. The student Minicomputer sets can be borrowed from a first, second, third, or fourth grade CSMP class, or 12 individual Minicomputer sheets and 6 sheets of checkers can be purchased from CSMP.

For the remainder of the lesson, have the students pair up and play Minicomputer Golf — one against the other.

25 N

"New" checkers may be made by the students as needed. Just turn a checker over and write a numeral on the back.

A flexible starting configuration is suggested below.



Goals could be 700 or 1,000 or any number the students choose.

If your class would enjoy it, a tournament could be organized and played.

5

DECIMALS

The students' first experiences with decimal numbers began in the CSMP Mathematics for the Upper Primary Grades and have been extended and broadened each subsequent semester. In this chapter, the review and extensions of these experiences occur in activities involving hand-calculator relations such as $\boxed{+}\,\boxed{0.2}\,\boxed{=}$ ... and $\boxed{+}\,\boxed{=}$ ..., graduations of portions of the number line, arrow roads with relations such as -2 and $+0.3$, decimal representations of numbers on the Minicomputer such as $25 = 24.999 \ldots = 24.\overset{*}{9}$, and completions of hand-calculator sentences.

CAPSULE LESSON SUMMARIES

Section 5.1:	Build roads from and to a dot for a given number using the relation $\boxed{+}\,\boxed{0.2}\,\boxed{=}$ ... and ending or starting at dots for numbers with specified properties. Build roads from and to a dot for a given number using the relation $\boxed{+}\,\boxed{=}$ ..., and identify the arrows in the form "$\square\times$". (Worksheets N35, N36, N37, N38)
Section 5.2:	For two given numbers located at points on the number line, find the number at the midpoint of the segment and find the numbers at the trisection points of the segment. Given a number line with two kinds of graduation points, where one graduation has its points labeled, determine the numbers at the points of the other graduation. (Worksheets N39, N40, N41, N42)
Section 5.3:	Build a shortest road from one to another of two given numbers, using arrows for two relations such as $3\times$ and $+0.2$. (Worksheets N43, N44, N45, N46)
Section 5.4:	Make plays on the Minicomputer to show that, for example, $2 = 1.999 \ldots = 1.\overset{*}{9}$. Complete sentences called "hand-calculator sentences" such as $\boxed{3.2}\,\boxed{}\,\boxed{5}\,\boxed{}\,\boxed{20}\,\boxed{=}\,\boxed{0.8}$, using the symbols: $+$, $-$, $\times$, $\div$, in the blank boxes. (Worksheets N47, N48, N49)

5N

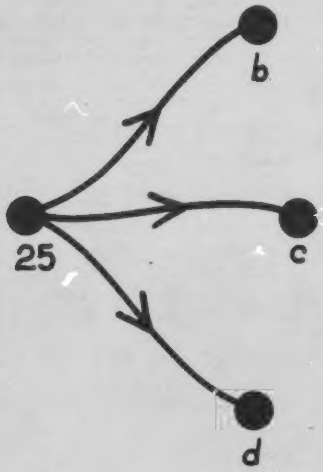
5.1 Hand-Calculator Relations with Decimals

Distribute as many hand-calculators as you have available.

Exercise 1

Draw this arrow picture on the board.

$\boxed{+}\boxed{0.2}\boxed{=}$*...



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: Start from 25 and press $\boxed{+}\boxed{0.2}\boxed{=}$... as many times as you like. What number could be here (point to b)?

Since we cannot predict what responses your students might make, we reproduce here dialogue that occurred in one of our development classes.

S: 25.6.

Label b "25.6".

---

* "Press $\boxed{+}\boxed{0.2}\boxed{=}$..." is an abbreviation for "press $\boxed{+}\boxed{0}\boxed{\cdot}\boxed{2}\boxed{=}$...". We will use this abbreviation where appropriate for the succeeding lessons in this guide.

T: How many times was $\boxed{=}$ pressed?

S: Three times.

T: What number could be here (point to c)?

S: 26.4.

Label c "26.4".

T: And how many times was $\boxed{=}$ pressed?

S: Seven times, because 7 × 0.2 = 1.4 and 25 + 1.4 = 26.4.

T: What number could be here (point to d)?

S: 45.0.

Label d "45.0".

T: How many times was $\boxed{=}$ pressed?

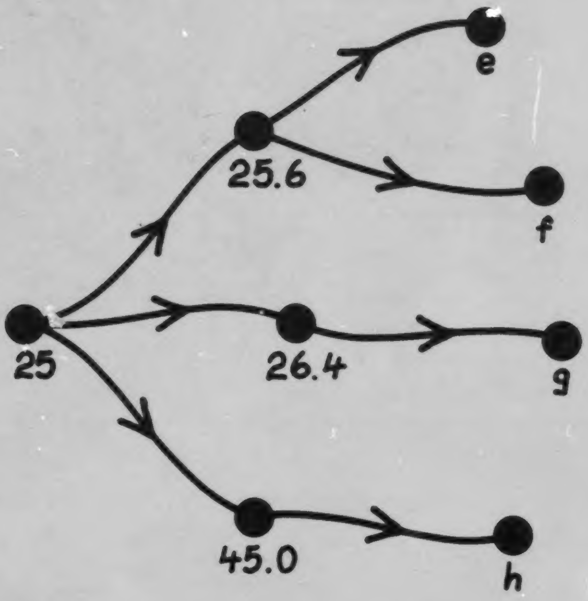
S: 100 times, because 100 × 0.2 = 20 and 25 + 20 = 45.

5 N

Add these arrows to your drawing.

$\boxed{+}\ \boxed{0.2}\ \boxed{=}$ . . .

Do not write the letters on the They are here just to make the description of the lesson easier to follow.



Continue the exercise by asking the following questions for e, f, g, and h. Be sure to label the dots with correct student responses.

at e: Which is the largest number smaller than 100 that could be here? (Answer: 99.8)

at f: Which is the smallest number larger than 75.5 that could be here? (Answer: 75.6)

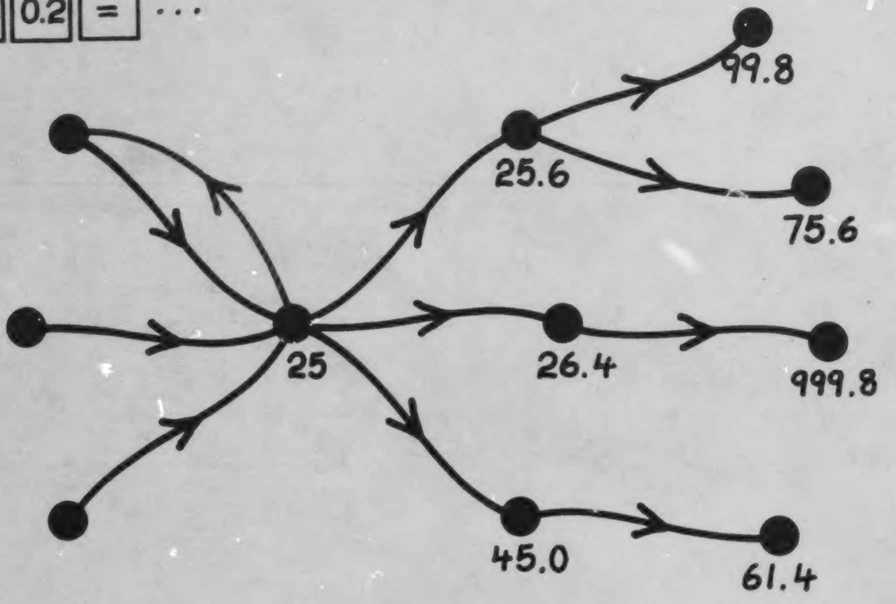
at g: Which is the largest number smaller than 1,000 that could be here? (Answer: 999.8)

at h: Which is the smallest number larger than 61.35 that could be here? (Answer: 61.4)

NOTE: It may happen that a student will respond that 61.36 is the number. Then you should remind the students that 0.2 is being added rather than 0.02.

Add these red arrows ending at 25 and a return blue arrow to your drawing.

$\boxed{+}\boxed{0.2}\boxed{=}$ . . .



T (tracing the blue arrow): What could this return arrow be for?

S: $\boxed{-}\boxed{0.2}\boxed{=}$ . . . .

T (pointing to the upper left dot): What number could be here?

Label the dot with any correct student response, e.g., 24.8; 20.0; 9.6; $^{-}3.8$; and so on.

Label the remaining two dots with

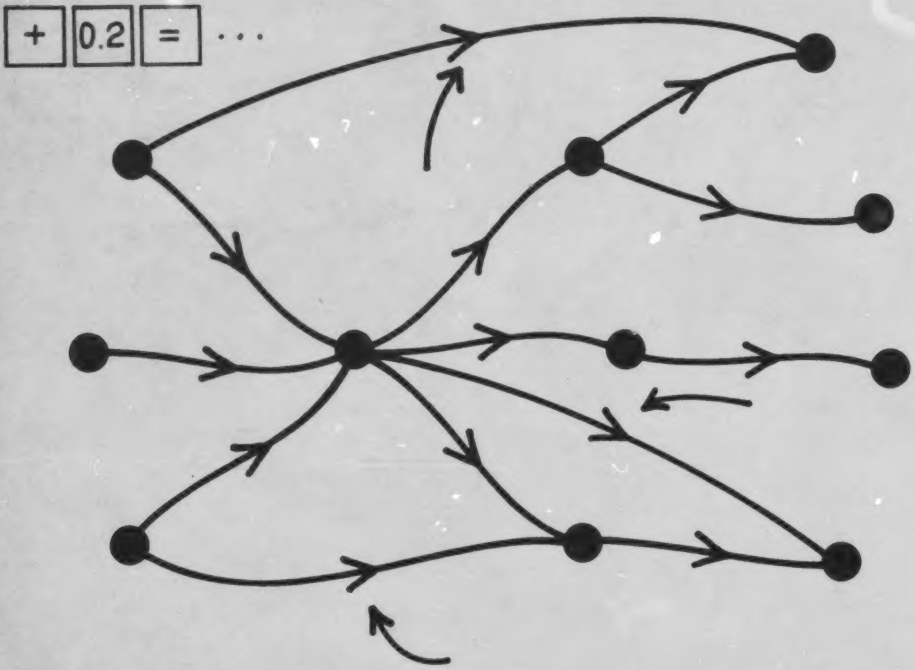
the largest number possible smaller than 15 (Answer: 14.8)
and
the smallest positive number. (Answer: 0.2)

Erase the numerals from the arrow picture.

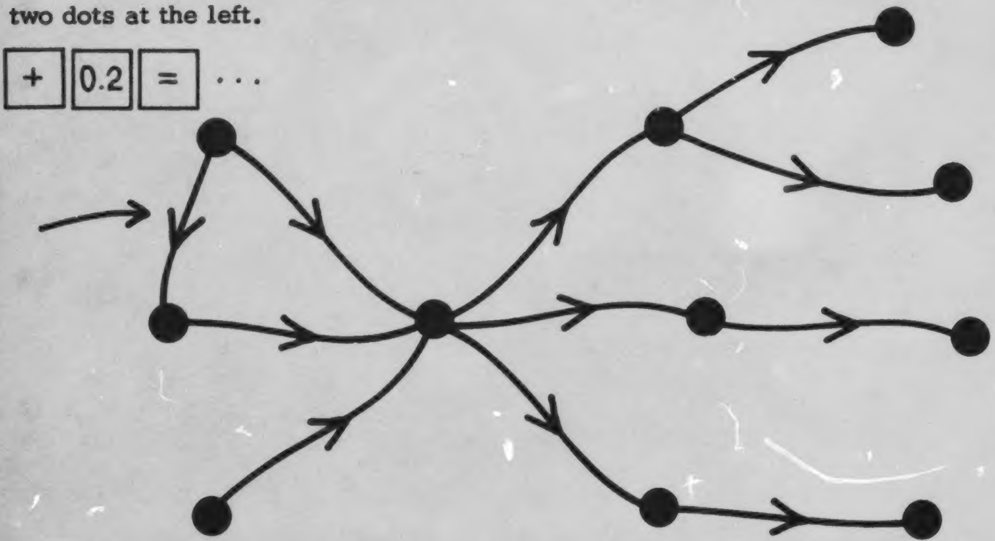
5 N

T: Can you trace some other red arrows that could be drawn in this picture?

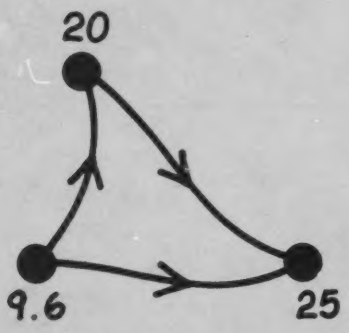
There are many possibilities; the picture below shows three examples.

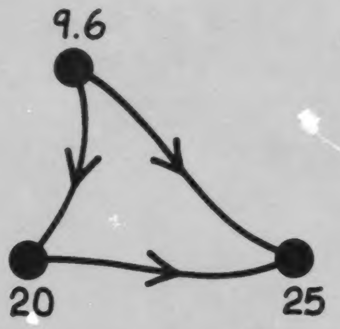


NOTE: It is possible that a student will suggest drawing an arrow between two dots at the left.



In fact an arrow may be drawn between any two beginning points or ending points; however, we cannot be sure of the direction of such an arrow. Consider two cases:

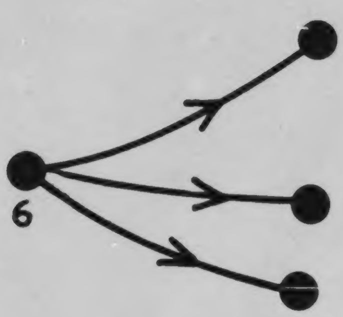




Exercise 2

Draw this picture on the board.

$\boxed{+}\boxed{=}$ . . .



T: Red arrows are for $\boxed{+}\boxed{=}\boxed{=}$ . . . as many times as you like. What numbers could be in the picture?

5N

Encourage the students to explore this situation using their hand-calculators. The discussion might develop along these lines.

S: 30 could be in the picture.

Label a dot "30".

T: Yes. How many times should we press $\boxed{=}$ to get 30?

S: Five times.*

Trace the arrow starting at 6 and ending at 30.

T: Could you label this arrow "$\boxed{}$ x"?

S: 5 x.

T: Good. What are some other numbers that could be in the picture?

Label the remaining two dots with correct student responses; e.g., 12; 144; 1,074; 90; or any multiple of 6 other than 6 and 30. Each time label the corresponding arrow "$\boxed{}$ x".

T: What do you notice about these numbers?

S: They are all even.

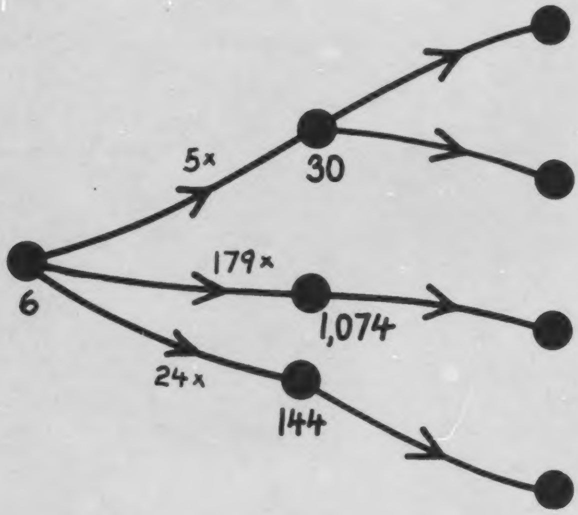
S: They are all multiples of 6.

---

* See the General Introduction Section 2(f) on the Role and Use of Hand-Calculators for a discussion of the $\boxed{+}$ $\boxed{=}$ ... relation.

Extend your picture by adding arrows to the right. Your picture should reflect the responses of your students and hence will be different though similar to this one.

$\boxed{+}\,\boxed{=}\ldots$



T: I have added some arrows to the picture. (Trace an arrow starting at 30.) What number could be here? Remember, we start from 30 and press $\boxed{+}\,\boxed{=}\ldots$.

S: 90.

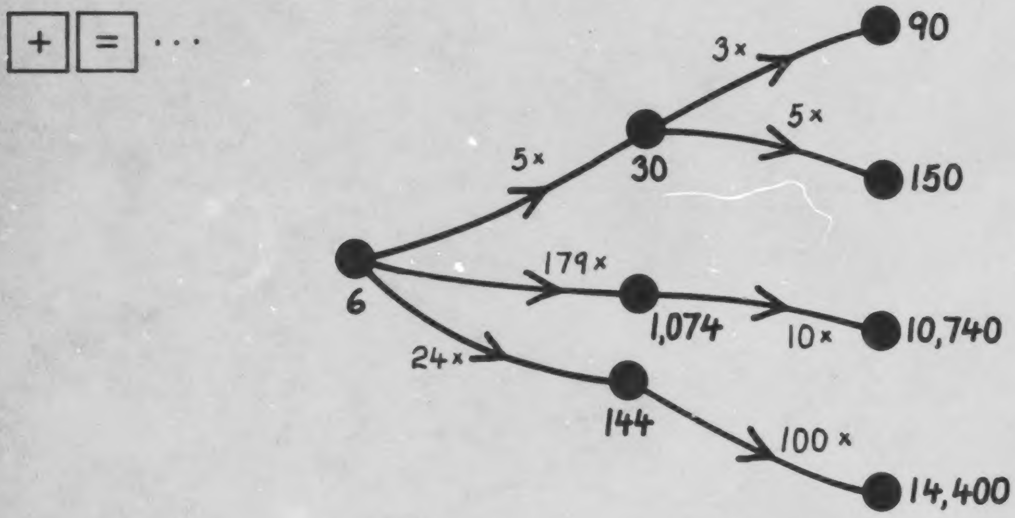
T: Very good. Could you label this arrow "$\boxed{}$ x"?

S: 3 x, because 3 x 30 = 90.

Label an arrow starting from 30 "3 x" and label its ending dot "90".

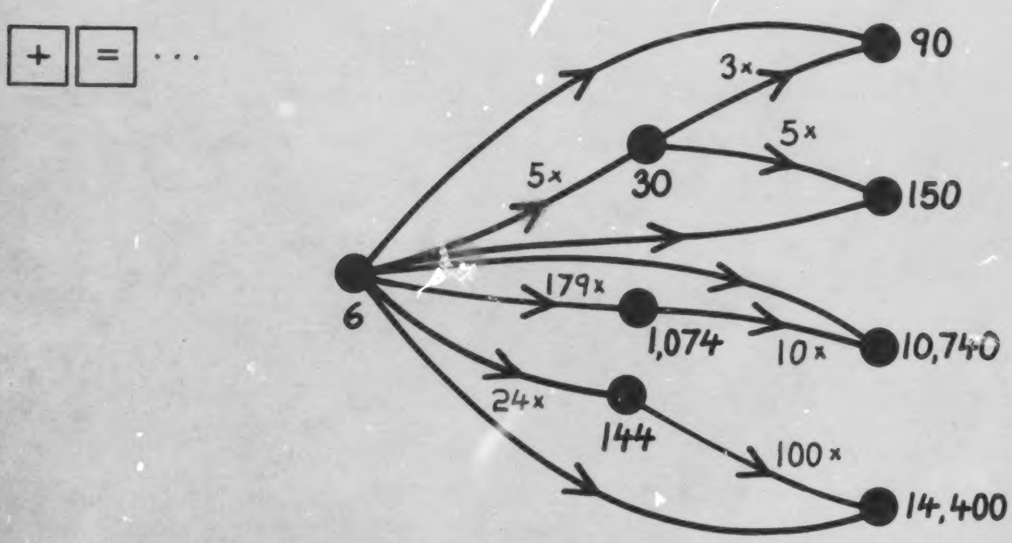
5 N

Continue in this manner labeling dots and corresponding arrows. After a while, your picture might look like this.


+
=
. . .
90
3×
5×
30
150
5×
6
179×
1,074
10×
10,740
24×
144
100×
14,400

T: Can you trace some other red arrows that could be drawn in our picture?

Invite several students to trace arrows. For the above picture, all of the additional red arrows that could be drawn are shown below unlabeled.


+
=
. . .
90
3×
5×
30
150
5×
6
179×
1,074
10×
10,740
24×
144
100×
14,400

For each additional arrow that was drawn in your picture, ask the class to label it "□x". For example,

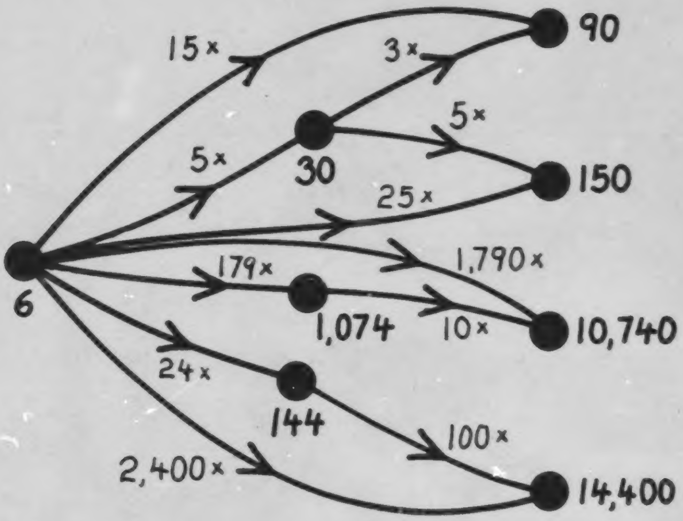
T (tracing the arrow starting at 6 and ending at 90): Could you label this arrow "□x"?

S: 15x.

T (tracing the appropriate arrows): 5x followed by 3x is the same as 15x.

An example of a completed arrow picture is shown below.

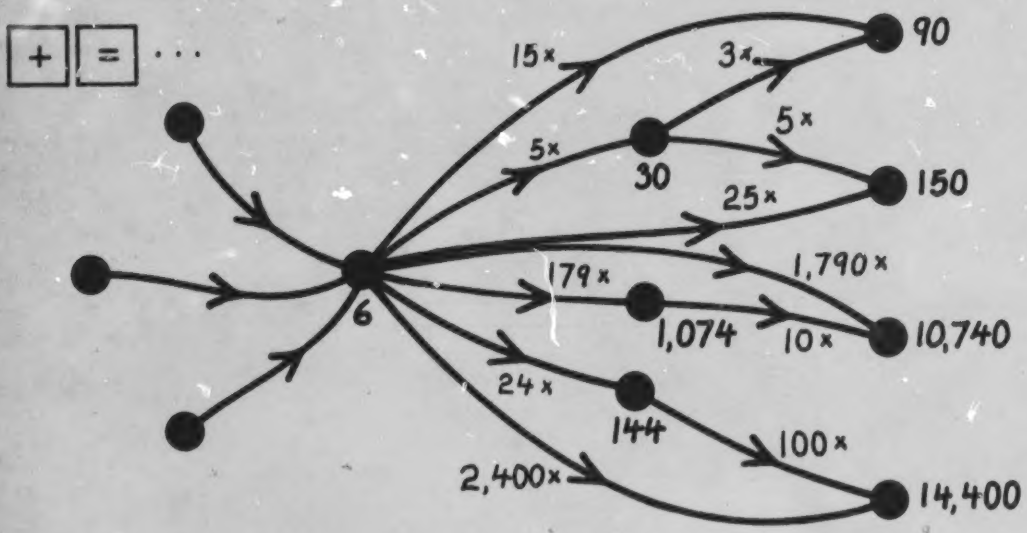
$\boxed{+}\ \boxed{=}$ . . .



15x
3x
90
5x
30
5x
150
25x
1,790x
179x
6
1,074
10x
10,740
24x
144
100x
2,400x
14,400

5 N

Add these arrows ending at 6 to the picture.



T (pointing to the new unlabeled dots): What numbers could be here?

The first suggestions are likely to be 1, 2, and 3. These are correct, but encourage the students to think of some (non-integer) decimal numbers; e.g., 1.5; 0.6; 0.25; 0.01; etc.

S: 0.6.

T: If you start with 0.6 on the display, how many times would you press $\boxed{=}$[*] to get 6?

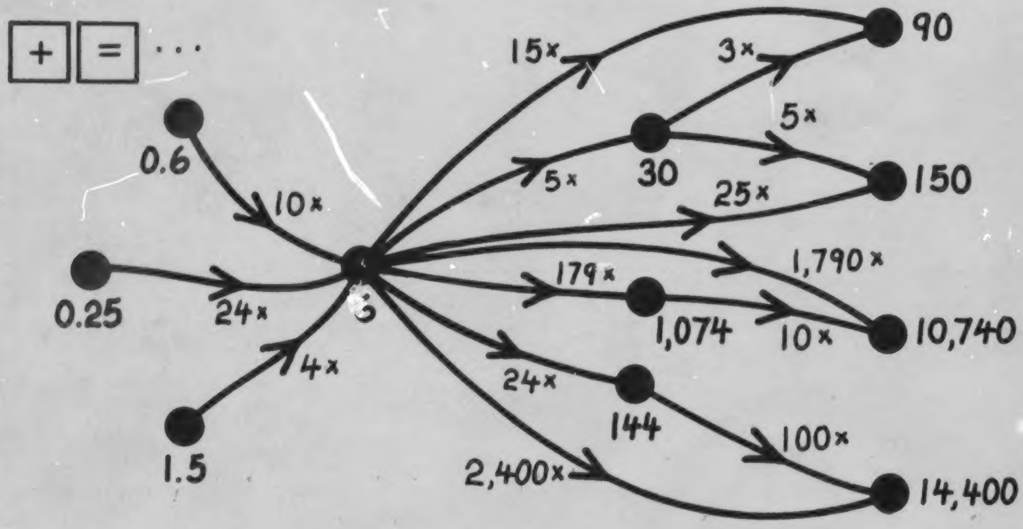
S: 10, because 10 × 0.6 = 6.

In this case, label a dot "0.6" and the corresponding arrow "10 x".

---

[*] See the General Introduction Section 2(f) on the Role and Use of Hand-Calculators for a discussion of the $\boxed{+}$ $\boxed{=}$ ... relation.

Continue the activity as long as there is student interest. A completed arrow picture is drawn below. Your arrow picture will probably be different since it should reflect the responses of your students.


+ = . . .
15×
3×
90
5×
30
5×
150
25×
0.6
10×
1,790×
0.25
24×
6
179×
1,074
10×
10,740
4×
24×
144
100×
1.5
2,400×
14,400

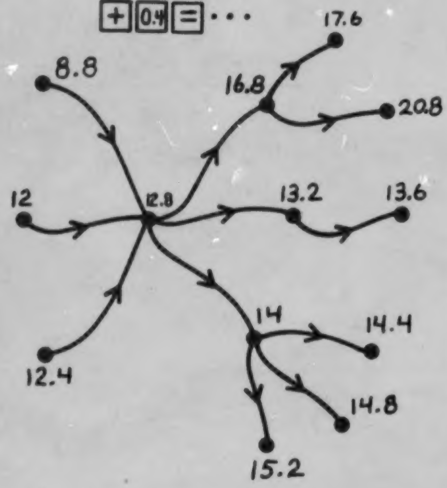
Worksheets N35, N36, N37, and N38 are available.

Name

N 35 *

Label the dots. Many answers are possible.

+ 0.4 = . . .


17.6
8.8
16.8
20.8
12
12.8
13.2
13.6
14
14.4
12.4
14.8
15.2

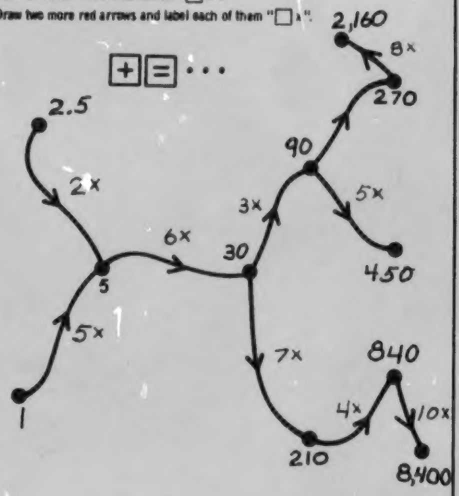
(Many other solutions are possible.)

Name

N 36 **

Label the dots. Label each arrow "☐x".
Draw two more red arrows and label each of them "☐x".

+ = . . .


2,160
8x
270
2.5
90
2x
5x
3x
6x
30
5
450
5x
7x
840
4x
10x
1
210
8,400

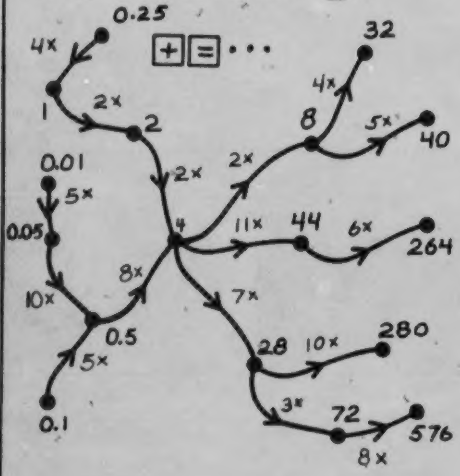
(Many other solutions are possible.)

Name

N 37 ***

Label the dots. Label each arrow "☐x".
Draw three more red arrows and label each of them "☐x".

+ = . . .


0.25
4x
32
1
2x
2
4x
8
5x
40
0.01
2x
2x
5x
0.05
4
11x
44
6x
264
8x
10x
7x
0.5
5x
28
10x
280
3x
72
0.1
576
8x

(Many other solutions are possible.)

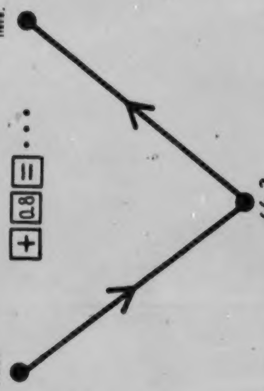
Name

N 38 ****

Kri is the smallest positive number which could be here.

+ 0.8 = . . .

Kra is the smallest number larger than 1,000 which could be here.



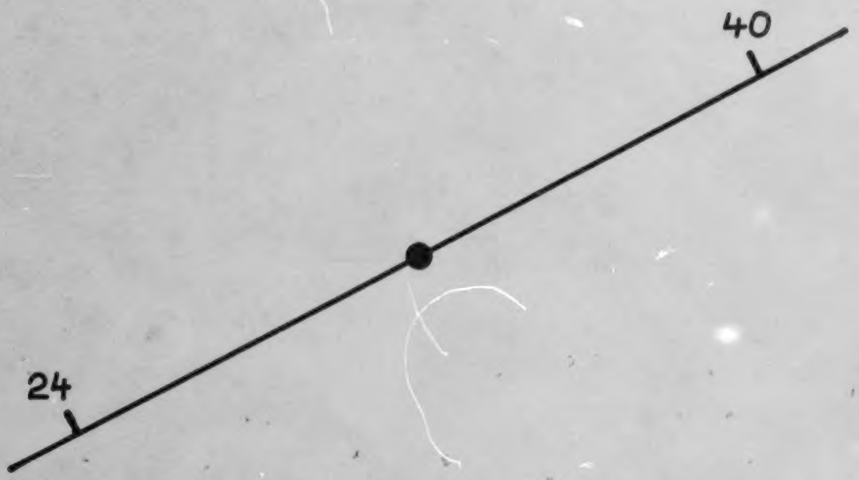
Kri is 0.6

Kra is 1,000.6

5.2 Decimals on the Number Line

Exercise 1

Draw this part of the number line on the board.



T: I have drawn a portion of the number line and have located 24 and 40. The dot is for the number that is the midpoint of 24 and 40. What is this number?

S: 32.

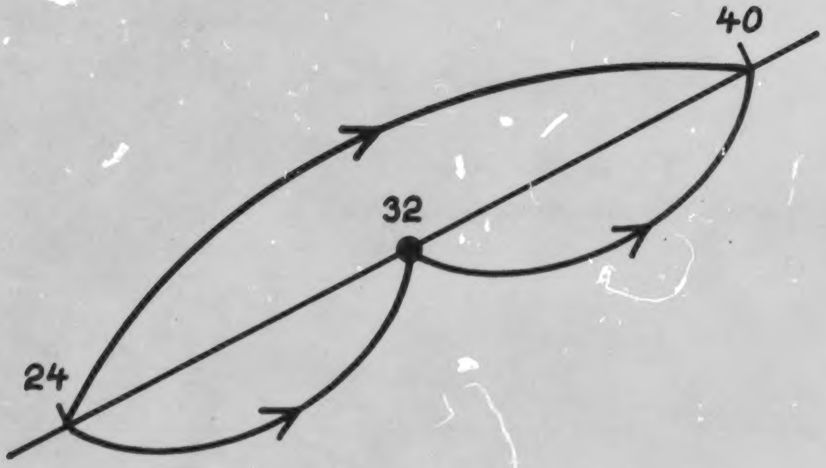
T: How can we check that 32 is the midpoint?

S: 32 is 8 more than 24 and is 8 less than 40.

T: Very good. Here is a way to picture the situation.

10 N

Add these arrows to the picture.



T: What could the red arrow be for?

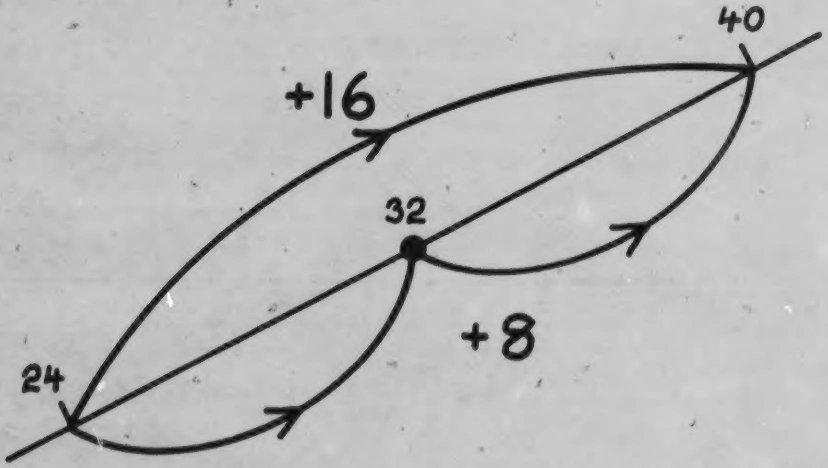
S: + 16.

Label the red arrow "+ 16".

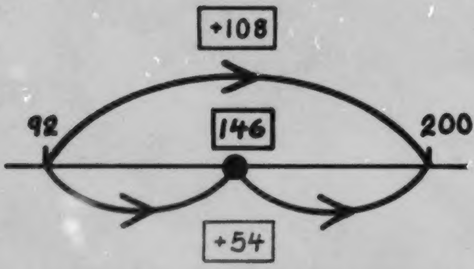
T: What could the blue arrows be for?

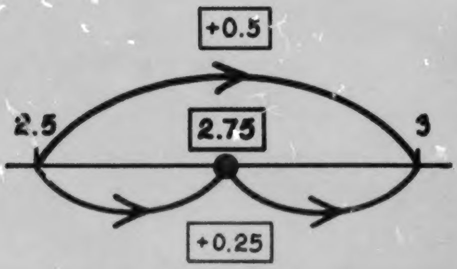
S: + 8.

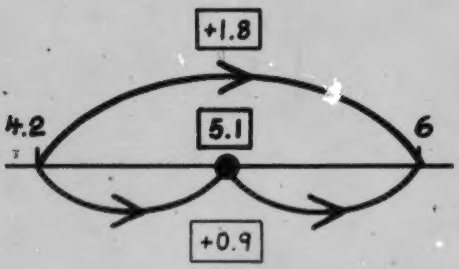
Label the blue arrows "+ 8". A completed picture is shown below.



Repeat this activity several times. Ask the students to write answers on their papers. Some suggestions for problems are illustrated below with answers in the boxes. Feel free to adjust the problems to the numerical level of your class.

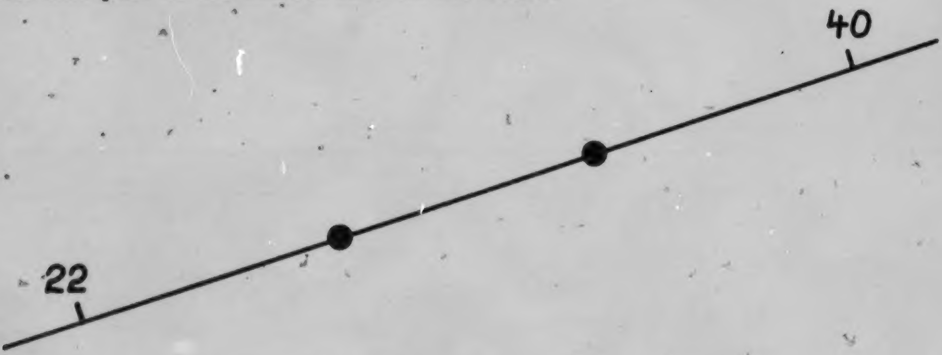

+108
92
146
200
+54


+0.5
2.5
2.75
3
+0.25



Exercise 2

Draw this part of the number line on the board.



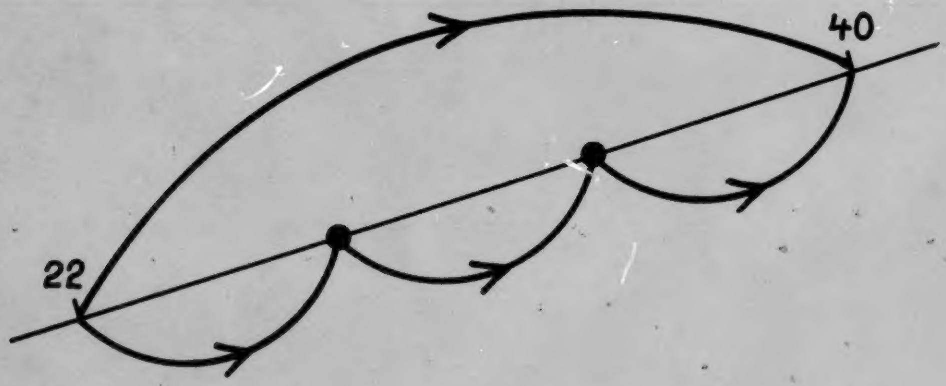
10 N

T: I have located 22 and 40 on this portion of the number line. The two dots divide the segment between 22 and 40 into thirds; the three segments are all the same length. Which numbers are these?

To aid the students in finding the numbers, you should suggest:

T: These arrows might help you solve this problem.

Add these arrows to your picture.



T: What could the red arrow be for?

S: + 18.

Label the red arrow "+ 18".

T: What could the blue arrows be for?

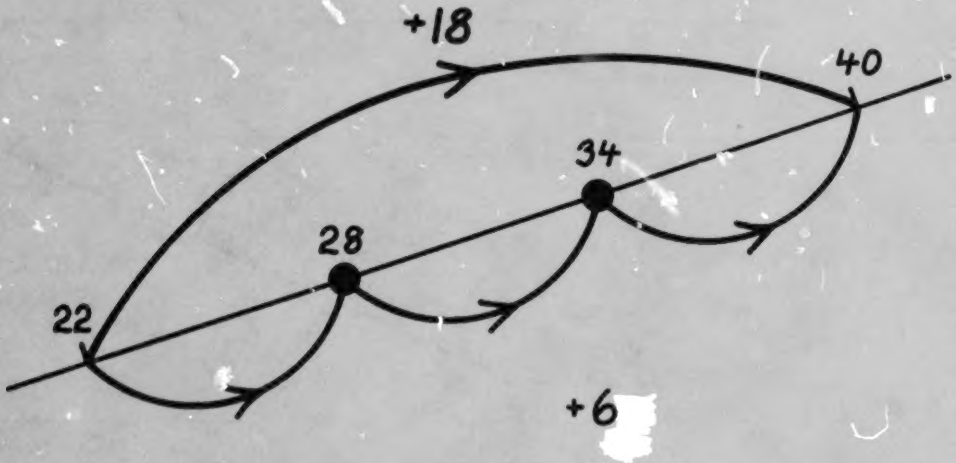
S: + 6.

Label the blue arrows "+ 6".

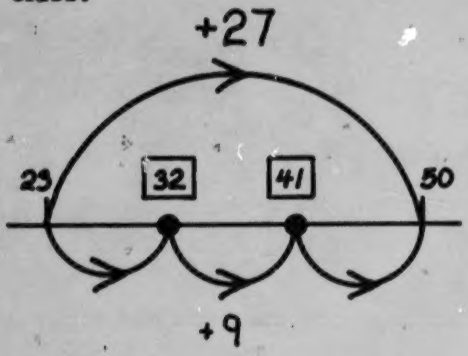
T: What are the two missing numbers?

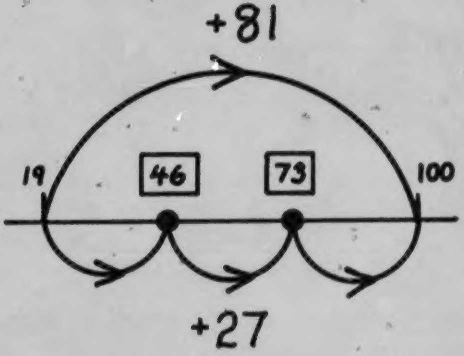
S: 28 and 34.

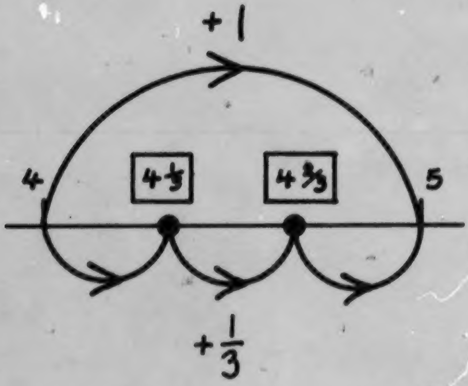
Label the two dots appropriately. A complete arrow picture is shown below.


+18
40
34
28
22
+6

Repeat this activity several times. Ask the students to write answers on their papers. Encourage those who have difficulty to use arrow pictures. Some suggested problems are illustrated below with appropriate arrows and boxed answers. Feel free to adjust these problems to the numerical level of your class.

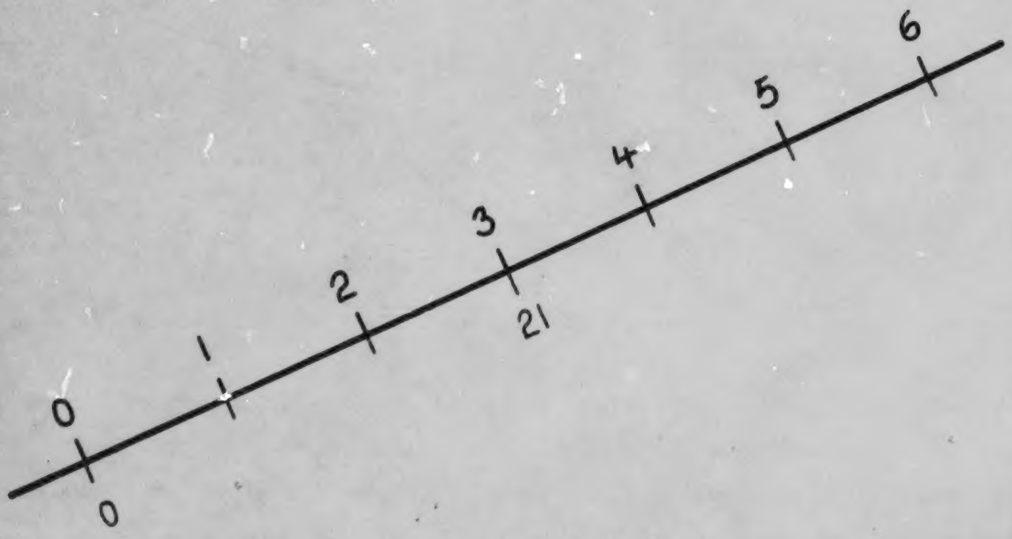

+27
23
32
41
50
+9


+81
19
46
73
100
+27



Exercise 3

Draw this picture on the board.



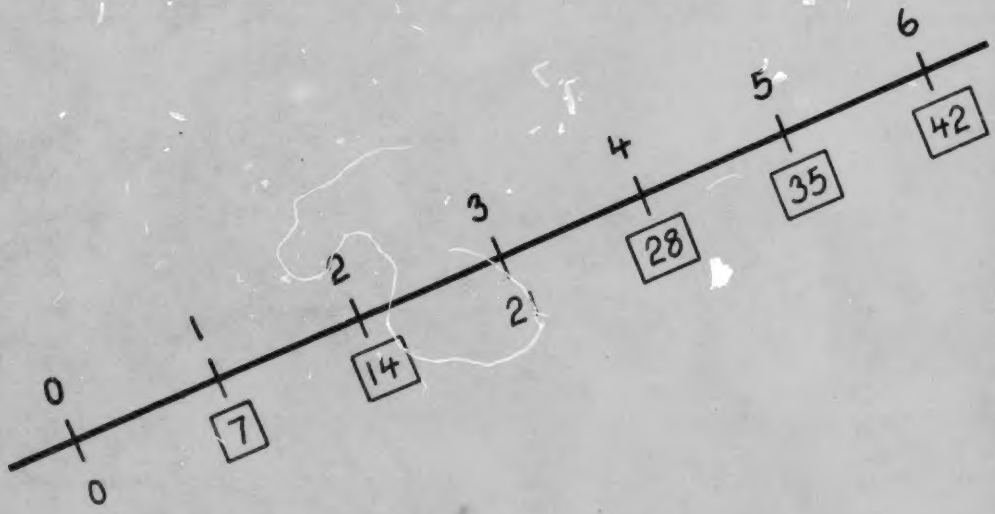
T: I have drawn a scale with two kinds of graduations. You have seen a ruler with inches on one side and centimeters on the other side. This scale is similar. You must complete the blue graduation.

Let the students have a minute or two to consider the problem.

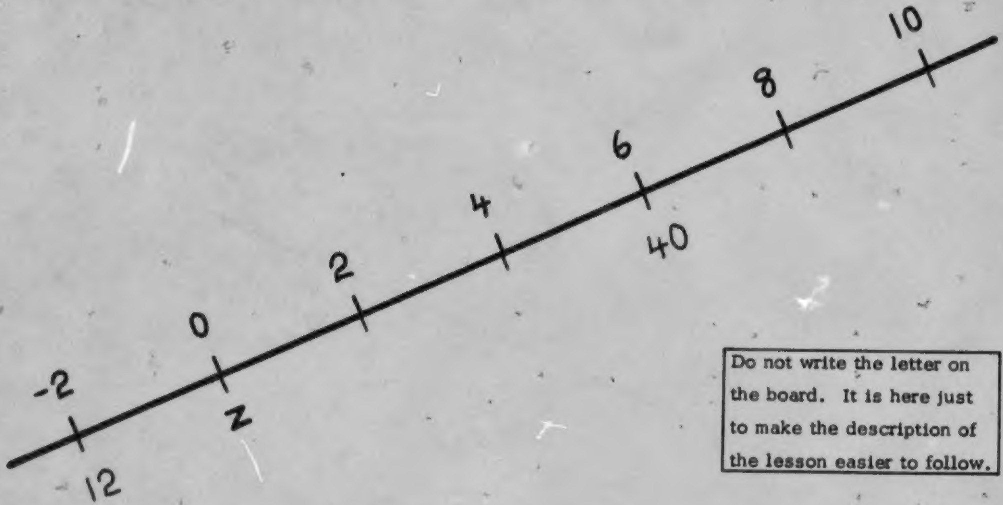
S: The segment between 0 and 21 is divided into three segments of the same length. 21 ÷ 3 = 7, so the numbers at each mark must be 7 more than the one before it.

S: Yes; 0, 7, 14, and so on.

A completed scale with correct answers in boxes is shown below.



Repeat the activity using this scale.



Do not write the letter on the board. It is here just to make the description of the lesson easier to follow.

T: Here the scale is slightly different; be careful. Complete the blue graduation.

For those students who have difficulty getting started, ask:

T: How much more is 40 than 12?

10 N

S: 28.

T: How many segments between 12 and 40?

S: 4.

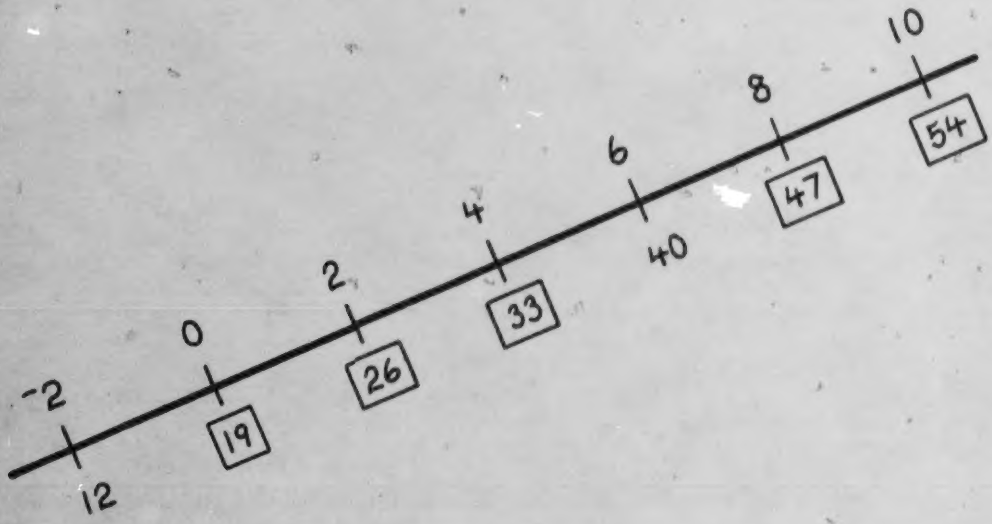
T (pointing to z): What number is here?

S: 19.

T: Can you explain?

S: From 12 to 40 is 28 and there are four segments of the same length, 28 ÷ 4 = 7 and 12 + 7 = 19.

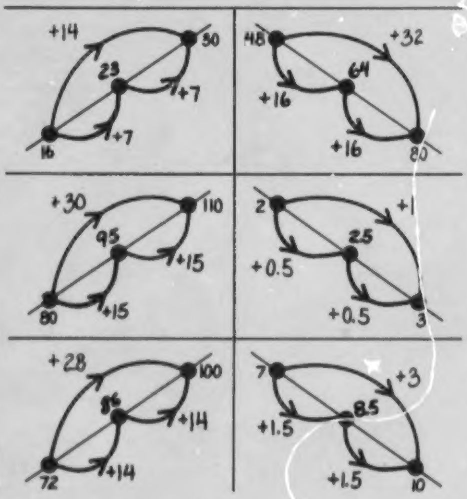
Continue labeling the blue graduation. A completed scale with correct answers in boxes is shown below.



Worksheets N39, N40, N41, and N42 are available.

Name ______________________ **N 39 ***

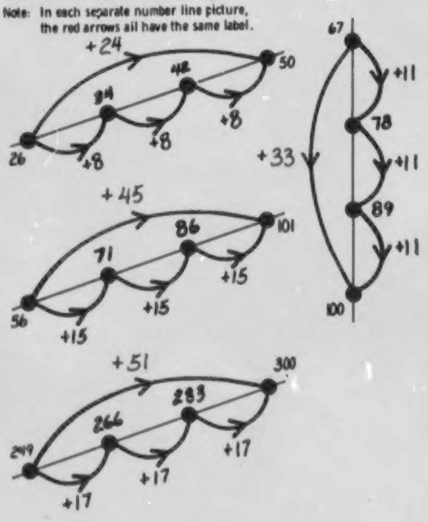
Label the midpoints. Label each arrow "+☐".


+14
23
50
+7
16
+7
48
+32
64
+16
+16
80
+30
95
110
+15
80
+15
2
+1
2.5
+0.5
+0.5
3
+28
86
100
+14
72
+14
7
+3
8.5
+1.5
+1.5
10

Name ______________________ **N 40 ****

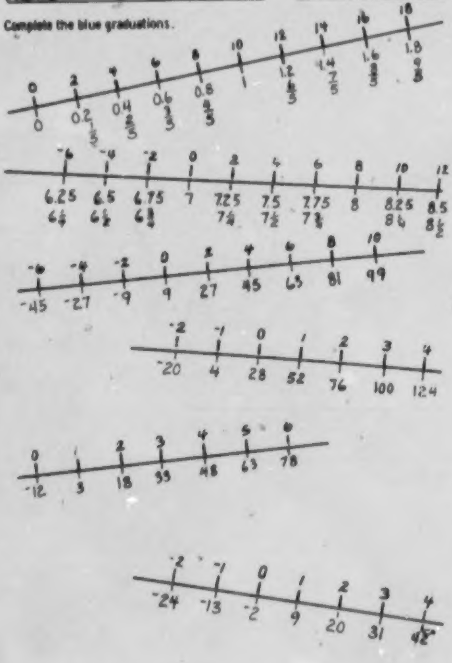
Label the dots. Label each arrow "+☐".

Note: In each separate number line picture, the red arrows all have the same label.


+24
50
42
34
+8
26
+8
+8
+8
+45
86
101
71
+15
56
+15
+15
+15
+51
300
283
266
+17
249
+17
+17
67
+11
78
+11
89
+11
+33
100

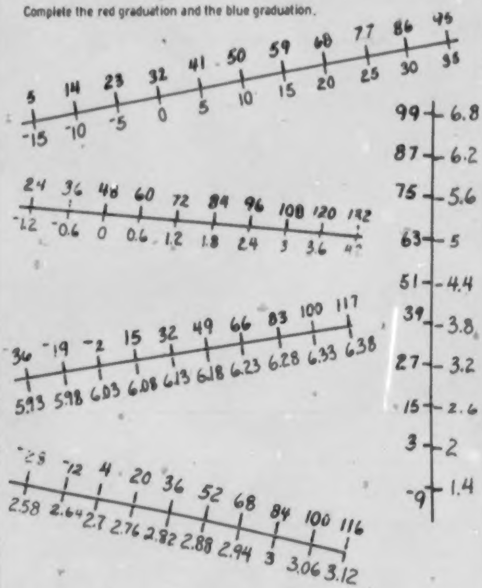
Name ______________________ **N 41 *****

Complete the blue graduations.


0 2 4 6 8 10 12 14 16 18
0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8
⁻6 ⁻4 ⁻2 0 2 4 6 8 10 12
6.25 6.5 6.75 7 7.25 7.5 7.75 8 8.25 8.5
⁻6 ⁻4 ⁻2 0 2 4 6 8 10
⁻45 ⁻27 ⁻9 9 27 45 63 81 99
⁻2 ⁻1 0 1 2 3 4
⁻20 4 28 52 76 100 124
0 1 2 3 4 5 6
⁻12 3 18 33 48 63 78
⁻2 ⁻1 0 1 2 3 4
⁻24 ⁻13 ⁻2 9 20 31 42

Name ______________________ **N 42 ******

Complete the red graduation and the blue graduation.

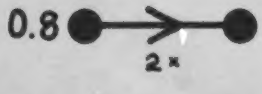

5 14 23 32 41 50 59 68 77 86 95
⁻15 ⁻10 ⁻5 0 5 10 15 20 25 30 35
24 36 48 60 72 84 96 108 120 132
⁻1.2 ⁻0.6 0 0.6 1.2 1.8 2.4 3 3.6 4.2
⁻36 ⁻19 ⁻2 15 32 49 66 83 100 117
5.93 5.98 6.03 6.08 6.13 6.18 6.23 6.28 6.33 6.38
⁻28 ⁻12 4 20 36 52 68 84 100 116
2.58 2.64 2.7 2.76 2.82 2.88 2.94 3 3.06 3.12
99 6.8
87 6.2
75 5.6
63 5
51 4.4
39 3.8
27 3.2
15 2.6
3 2
⁻9 1.4

5.3 Arrow Roads with Decimals

Exercise 1

T: I shall draw an arrow road on the board and you will label the dots or fill in the boxes for the arrows.

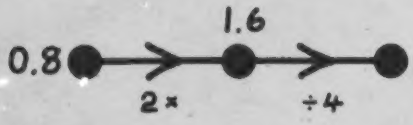
Start the road with this arrow.



T: What number is here (point to the ending dot)?

S: 1.6.

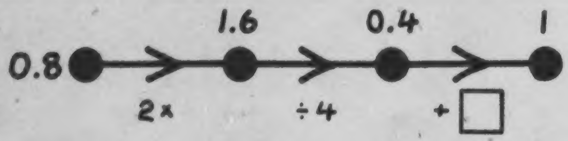
Label the dot "1.6" and add this arrow to your drawing.



T: What number is here (point to the ending dot)? Write it on your paper.

S: 0.4.

Label the dot "0.4" and add this arrow to your drawing.

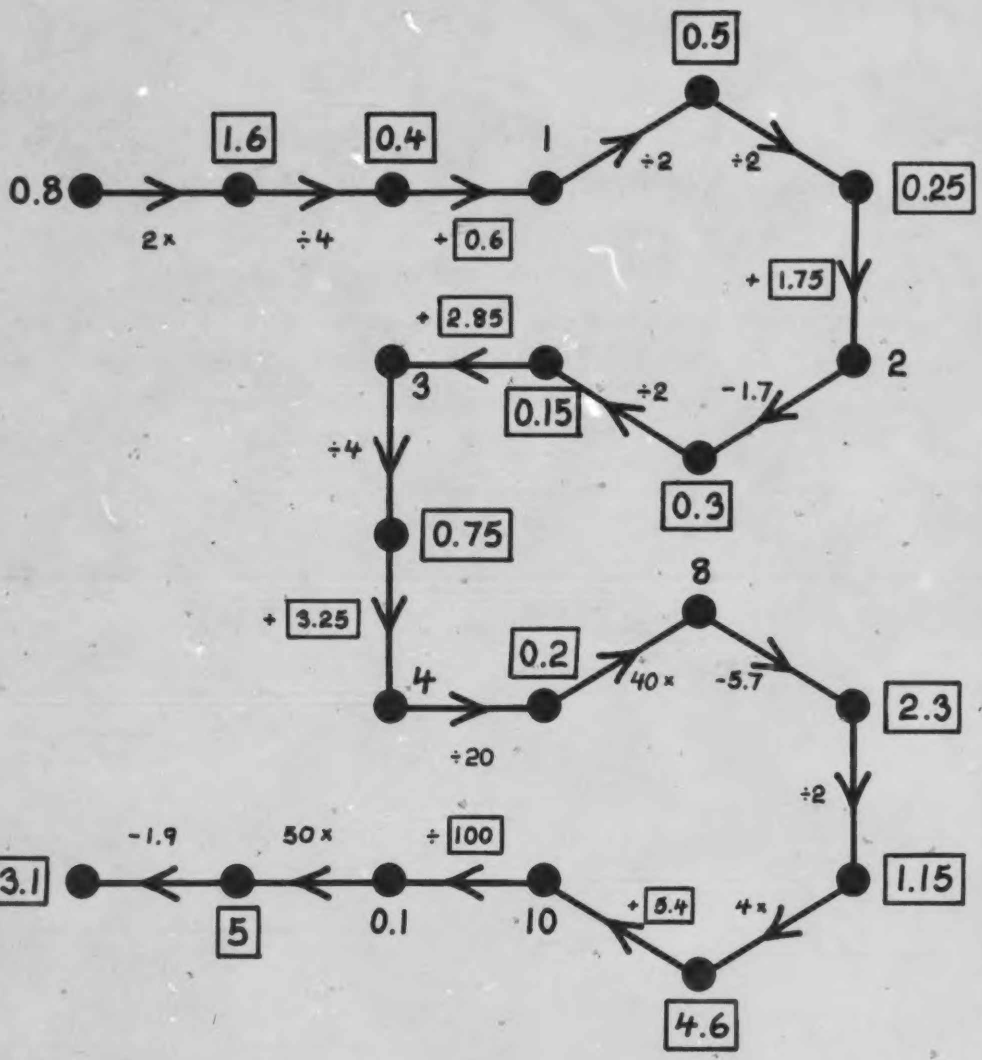


2,7

T: What could this arrow be for ?

S: +0.6.

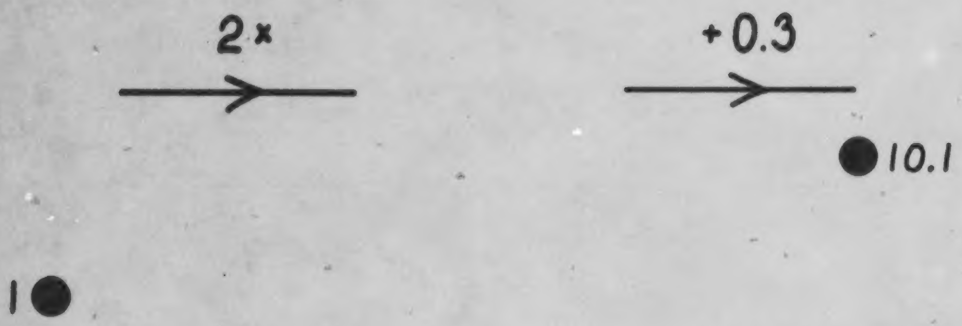
Continue extending and labeling this arrow road as long as there is student interest. The arrow road below was drawn in one of our development classes. Correct student responses are in the boxes.


0.5
1.6
0.4
1
÷2
÷2
0.8
0.25
2×
÷4
+0.6
+1.75
+2.85
3
0.15
÷2
-1.7
2
÷4
0.3
0.75
8
+3.25
0.2
4
40×
-5.7
2.3
÷20
÷2
-1.9
50×
÷100
1.15
3.1
5
0.1
10
+5.4
4×
4.6

13 N

Exercise 2

Put this information on the board.

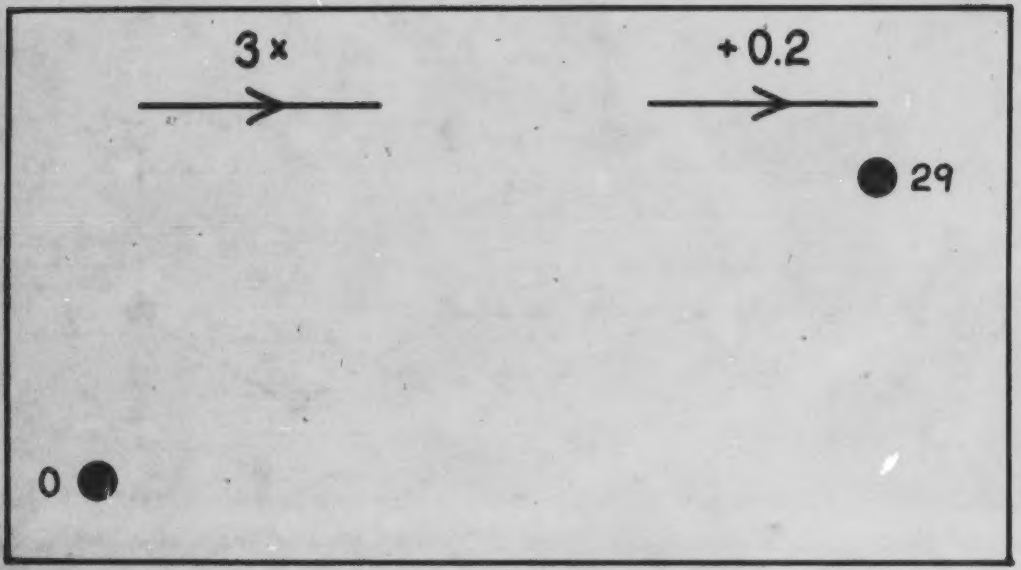


T: Build a red-blue road from 1 to 10.1. Try to use as few arrows as possible. See if you can use fewer than eight arrows.

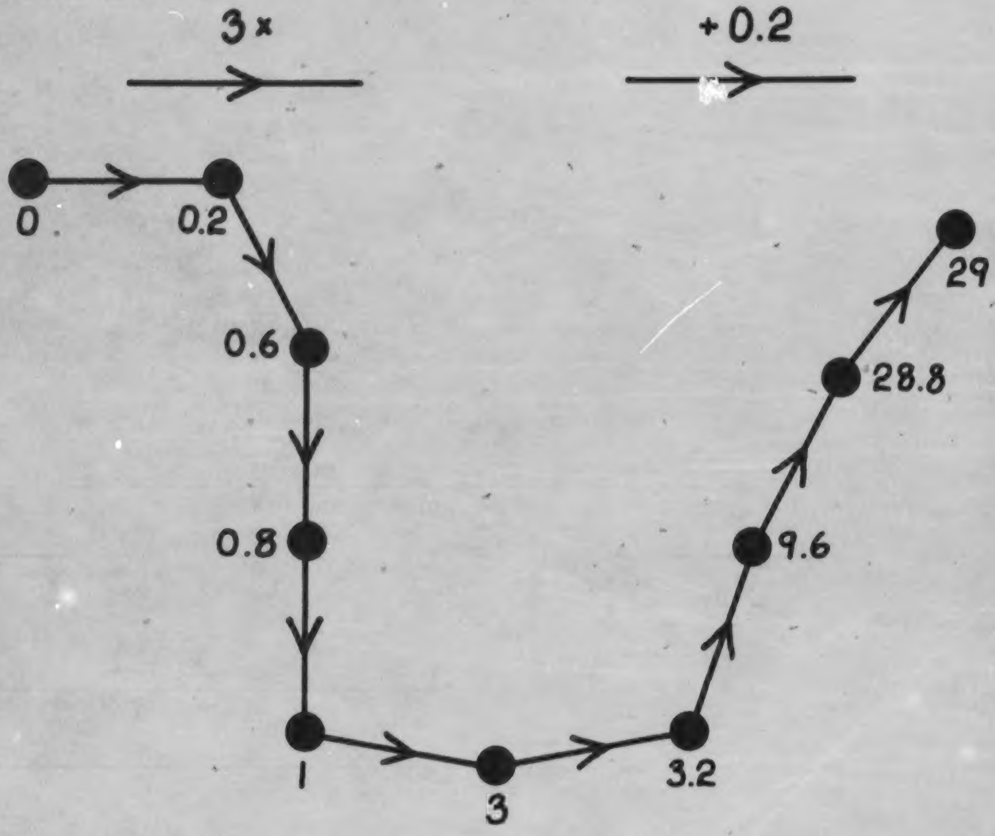
Let the students have some time to work independently. To create a further challenge to the class, let students who find solutions put the number of arrows they used on the board. Encourage the students to "beat the number" on the board. A road with six arrows is the shortest possible road.

Let the faster workers solve this three-star arrow road problem.

***



A solution to this three-star problem using the fewest number (nine) of arrows is shown below.



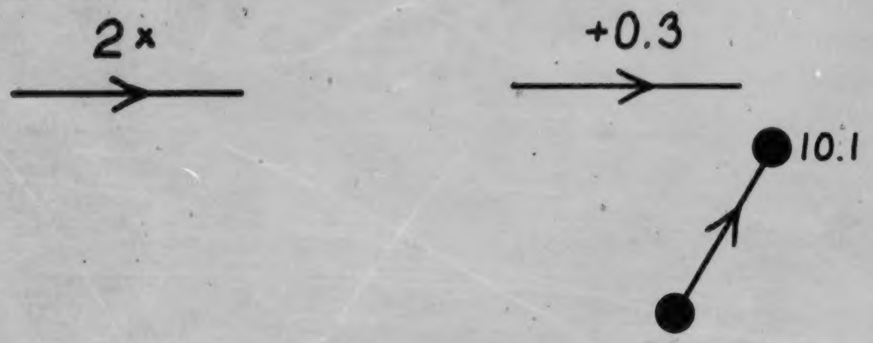
When it is obvious that many students have found at least one solution to the first problem, continue the collection discussion.

T: Many of you have found at least one arrow road. Let's look at an easy way to build the shortest road. Rather than start at 1, suppose we look first at an arrow ending at 10.1.
Which arrow - red or blue - do you think we should use?

S: Blue.

13 N

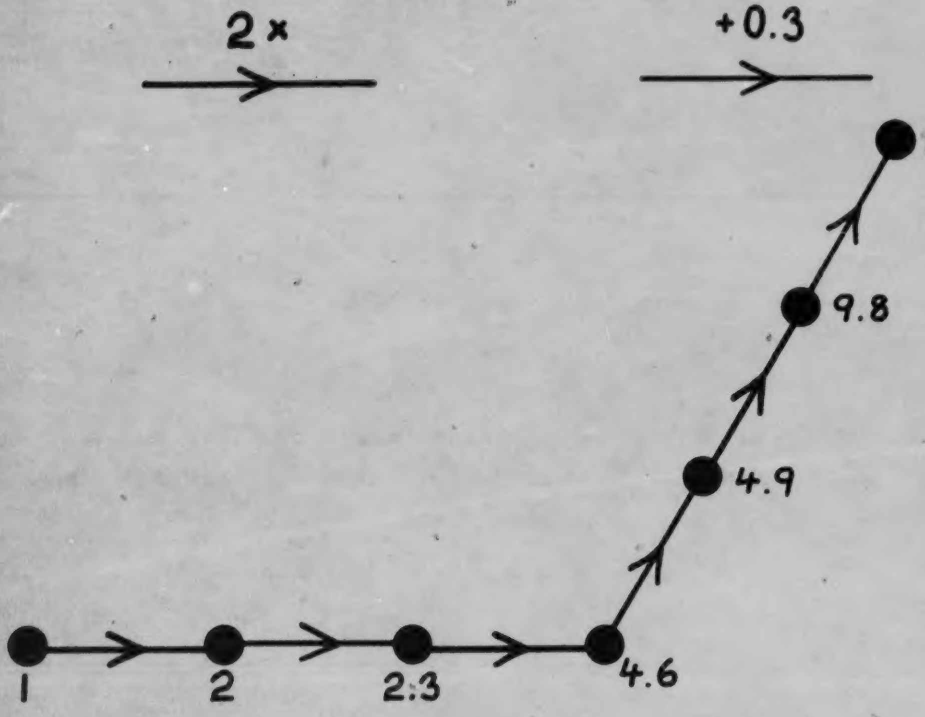
Draw this blue arrow.



T (pointing to the starting point of the blue arrow): What number is here?

S: 9.8.

Label the dot "9.8". Continue in this manner until the road is complete. The shortest road (six arrows) is shown below.


2×
+0.3
10.1
9.8
4.9
1
2
2.3
4.6

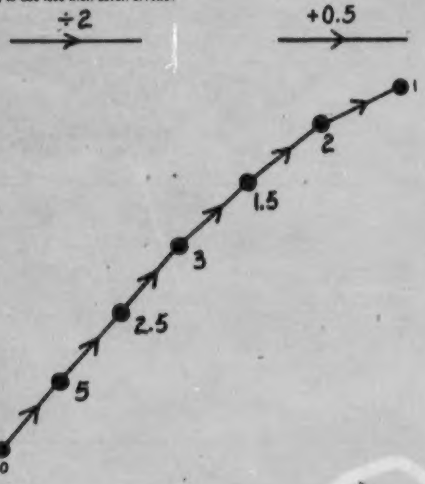
If time and interest permit, the three-star problem could also be solved collectively in this manner.

Worksheets N43, N44, N45, and N46 are available.

Name ____________________

N 43 *

Build a blue - red road from 10 to 1.
Try to use less than seven arrows.

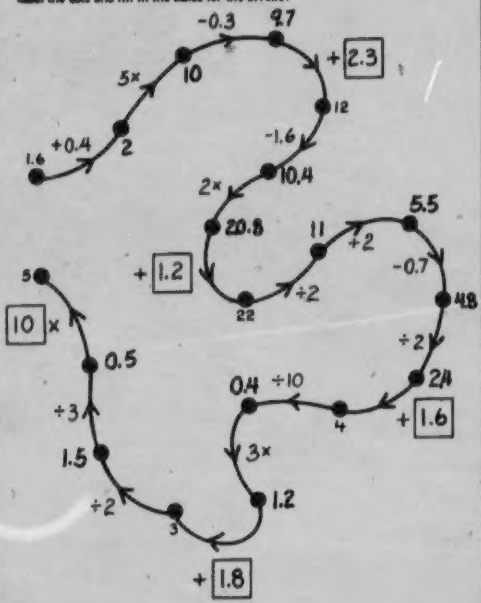

÷2
+0.5
1
2
1.5
3
2.5
5
10

(Other solutions are possible but this is the only one with less than seven arrows.)

Name ____________________

N 44 **

Label the dots and fill in the boxes for the arrows.

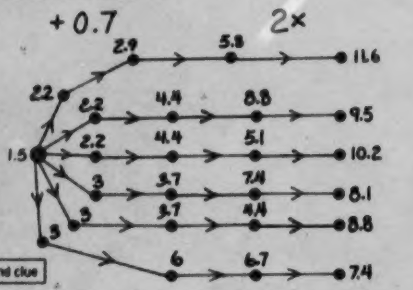

-0.3
9.7
+ 2.3
5×
10
12
+0.4
2
1.6
-1.6
10.4
2×
20.8
5.5
11
+2
-0.7
+ 1.2
÷2
22
5
10 ×
4.8
÷2
0.5
0.4
÷10
2.4
÷3
4
+ 1.6
3×
1.5
1.2
÷2
3
+ 1.8

Name ____________________

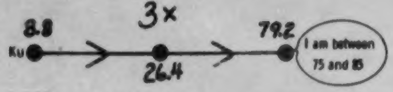
N 45 ***

First clue

Ku is the ending number of a red - blue road starting at 1.5 and using exactly two red arrows and two blue arrows.


+ 0.7
2×
2.9
5.8
11.6
2.2
2.2
4.4
8.8
9.5
1.5
2.2
4.4
5.1
10.2
3
3.7
7.4
8.1
3
3.7
4.4
8.8
3
6
6.7
7.4

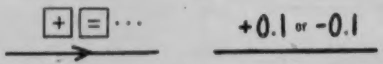
Second clue


3×
8.8
Ku
26.4
79.2
I am between 75 and 85

Ku is 8.8.

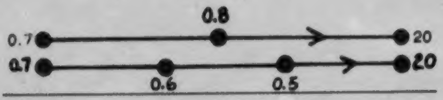
Name ____________________

N 46 ****

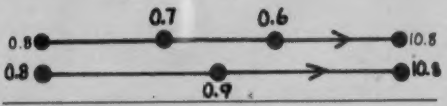


Use at most three arrows or cards to build a red - blue road

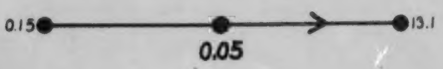
from 0. 7 to 20.


0.8
0.7
20
0.7
20
0.6
0.5

from 0. 8 to 10. 8.


0.7
0.6
0.8
10.8
0.8
10.8
0.9

from 0. 15 to 13. 1.

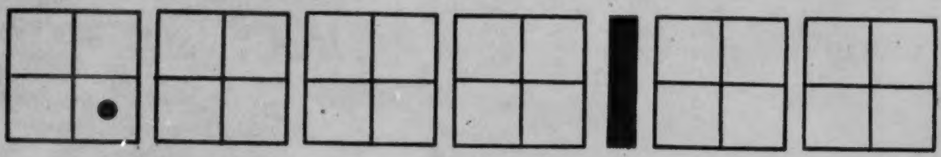


5.4 Decimals on the Minicomputer

Exercise 1

Display six Minicomputer boards and one positive regular checker.

NOTE: Only four Minicomputer boards are supplied in the Demonstration Minicomputer Set that is part of the instructional materials for this semester. Therefore, you will need to draw (on the chalkboard) two boards either to the right or left of the four that are provided, or if possible, borrow two boards from another teacher who is using CSMP materials.

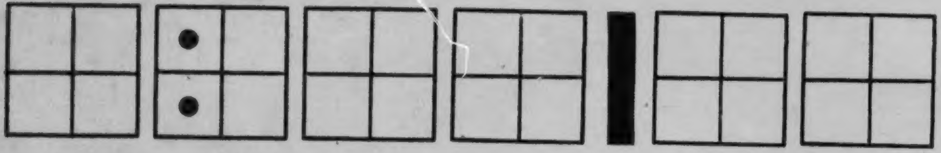


T: Do you remember the story of "1,000's Dream"?

Invite the students to recall as much of the story as they can.

T: Each morning 1,000 exercises on his Minicomputer. (Point to the thousands' board.) Here is 1,000. Then he jumps.

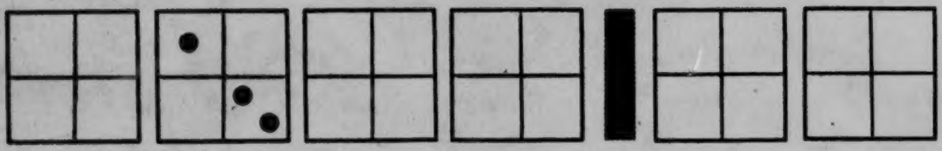
Make the jump on your boards.



32 N

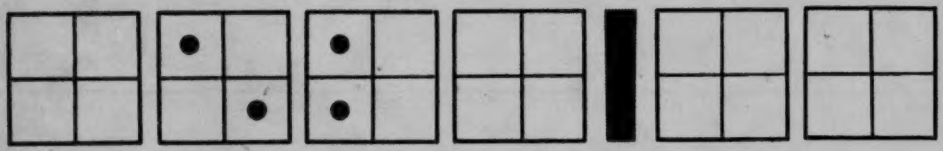
T: Then he jumps again.

Make the jump resulting in this configuration.

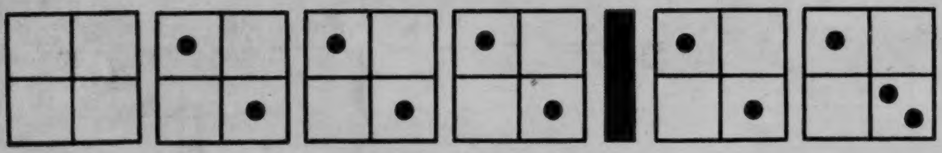


T: Who can make the next jump?

Let a student come to the board and make the jump resulting in this configuration.



Let the students continue to make jumps until the Minicomputer looks like this.



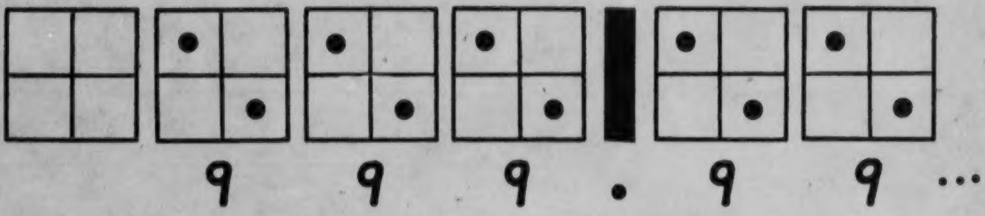
T: Could 1,000 keep jumping?

S: Yes, on and on, forever.

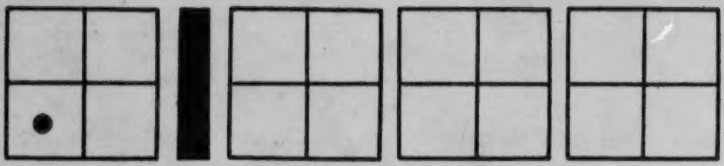
T: Do you remember how 1,000 wrote his name?

S: 999.999999....

Write this name for 1,000 below (above) your Minicomputer boards.



Display this configuration.

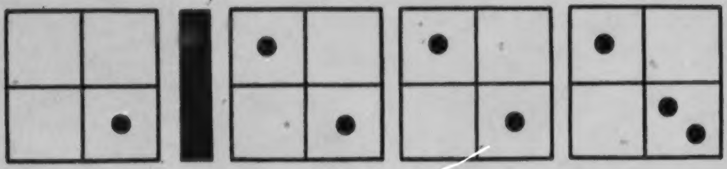


T: Do you think the number 2 could have the same dream and jump the same way?

S: Yes.

32 N

Let students make the jumps to obtain this configuration.

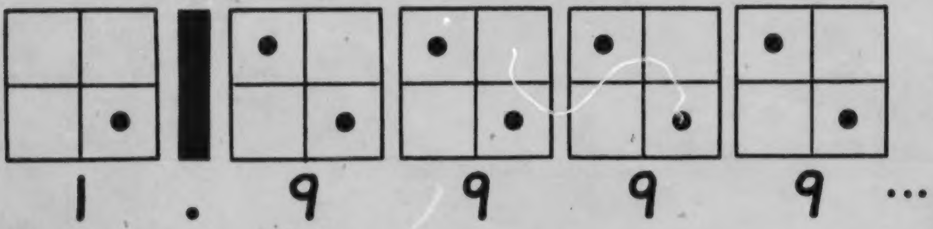


T: Imagine many more Minicomputer boards to the right and that 2 continues jumping. Write on your paper a new name for 2.

Walk around the room and observe the students' reactions to this situation.

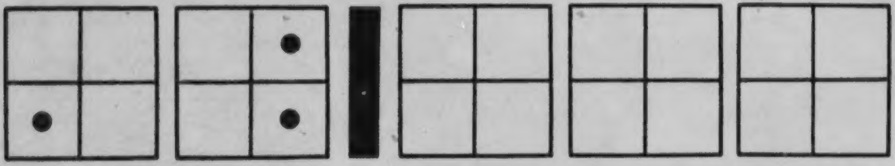
S: 2 = 1.999....

Write this name for 2 below (above) your Minicomputer boards.



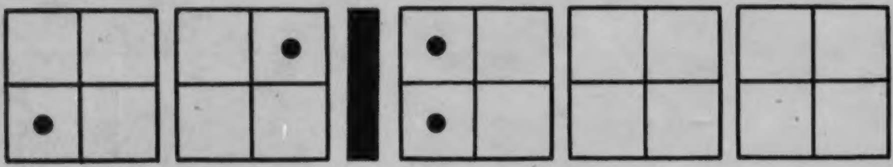
Display this configuration.

NOTE: If you did not borrow extra boards, it will be necessary to draw one Minicomputer board to the left of the four that are provided in the Demonstration Minicomputer Set.

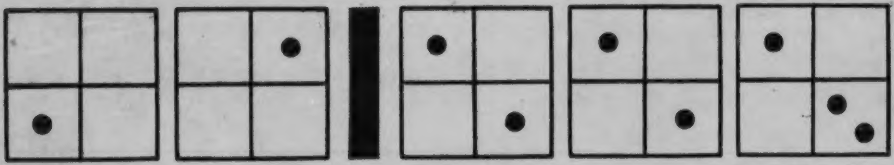


T: What would happen if 25 had this dream?

Let students make the jumps. You may have to suggest the first jump yourself.



Then the students should continue making jumps until the configuration shown below is reached.



T: Of course, we could continue jumping if we had more Minicomputer boards. Write a new name for 25.

Write this sequence of equalities on the board.

$$25 = 24.999\ldots = 24.\overset{*}{9} = 24.9\overset{*}{9} = 24.99\overset{*}{9}$$

32 N

T: Look carefully at what I have written. I put a little star over the "9" to show that the "9"'s go on and on.
Write a new name for 376.

S: $376 = 375.\overset{*}{9}$.

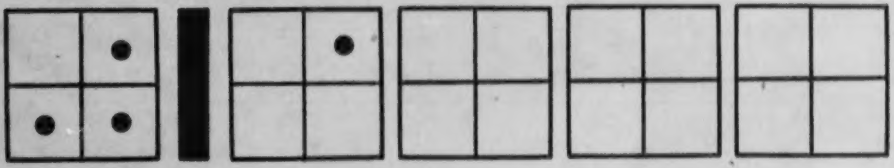
S: $376 = 375.999\ldots$.

Write these equalities on the board.

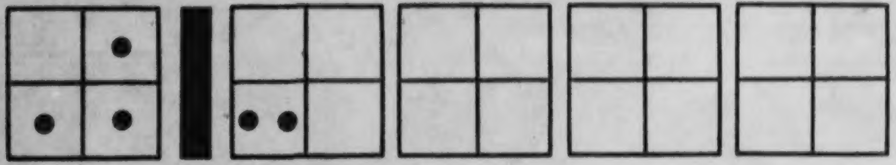
$$376 = 375.999\ldots = 375.\overset{*}{9}$$

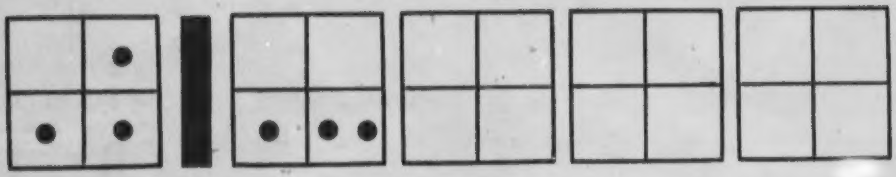
T: Write another name for 7.4.

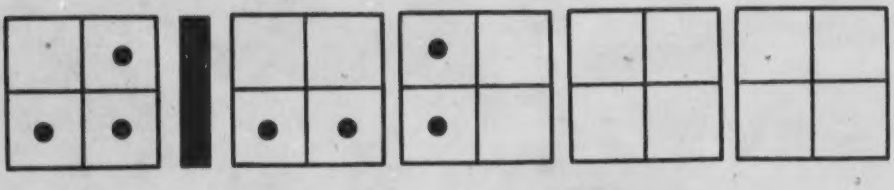
Some students might find this difficult. If that is the case, put 7.4 on the Minicomputer.

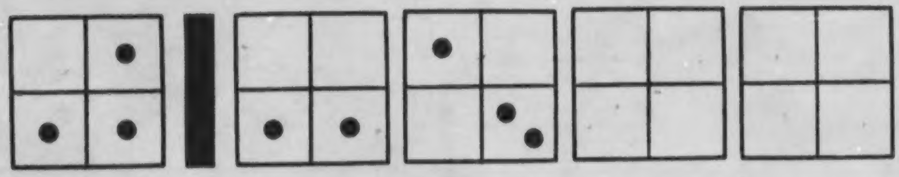


Let students make the jumps. The following sequence of illustrations shows the jumps that should be made.

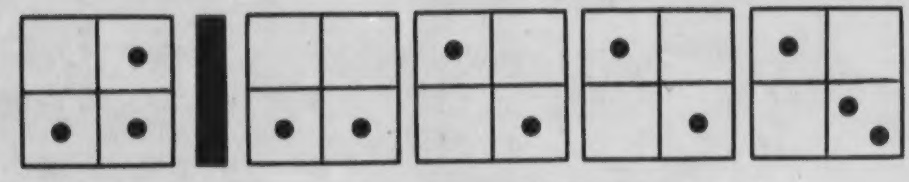








. . . and so on, until you obtain this configuration.



T: What is another name for 7.4?

S: 7.3999....

S: $7.3\overset{*}{9}$.

NOTE: Here it is very important that the "*" is not over the "3".

32 N

Exercise 2

Distribute as many hand-calculators as are available. If you have access to only a few hand-calculators, ask students to share or give the hand-calculators to those students who are having the most difficulty with computations.

Write these problems on the board.

$$\begin{aligned} 2 \times \boxed{} &= 100 \\ 4 \times \boxed{} &= 100 \\ 8 \times \boxed{} &= 100 \\ 16 \times \boxed{} &= 100 \\ 32 \times \boxed{} &= 100 \\ 64 \times \boxed{} &= 100 \end{aligned}$$

T: Solve these problems. You may use your hand-calculator to help you.

Let the students work independently for a moment or two; then complete the first three problems collectively.

$$\begin{aligned} 2 \times \boxed{50} &= 100 \\ 4 \times \boxed{25} &= 100 \\ 8 \times \boxed{12.5} &= 100 \end{aligned}$$

T (pointing successively to "2", "4", "8", "16", "32", and "64"): Do you notice anything interesting here?

S: Each number is the double of the one before it.

T: What do you notice about the numbers in the boxes?

S: Each number is one-half of the number before it.

T: Continue your work. 16 times what number is 100?

S: 6.25; $\frac{1}{2} \times 12.5 = 6.25$.

T: What about the next calculation?

S: $6.25 = 6.250$ and $\frac{1}{2} \times 6.250 = 3.125$, so $32 \times 3.125 = 100$.

T: What about this last calculation?

S: We must find $\frac{1}{2} \times 3.125$.

If there are some students who have difficulty with this calculation, ask:

T: What is $\frac{1}{2} \times 3$?

S: 1.5.

Write "$\frac{1}{2} \times 3 = 1.5$" on the board.

T: What is $\frac{1}{2} \times 0.12$?

S: 0.06.

Write "$\frac{1}{2} \times 0.12 = 0.06$" on the board.

T: $0.125 = 0.1250$. What is $\frac{1}{2} \times 0.1250$? Write it on your paper.

Write the problem on the board.

32 N

$$0.125 = 0.1250$$
$$\frac{1}{2} \times 0.1250 = \boxed{}$$

S: 0.0625.

Complete the calculation on the board.

$$\frac{1}{2} \times 0.1250 = \boxed{0.0625}$$

Bring the following information together on the board.

$$\frac{1}{2} \times 3.0 = 1.5$$
$$\frac{1}{2} \times 0.1250 = 0.0625$$

$$\frac{1}{2} \times 3.1250 = \boxed{}$$

T: Complete the calculation on your paper.

S: Just add. $\frac{1}{2} \times 3.1250 = 1.5625$.

S: So $64 \times 1.5625 = 100$.

Exercise 3

Write this hand-calculator sentence[*] on the board.

$$\boxed{7}\ \boxed{}\ \boxed{2.5}\ \boxed{}\ \boxed{4}\ \boxed{=}\ \boxed{18}$$

T: Each blank box may be filled with one of these symbols: +, x, -, ÷ to make the hand-calculator sentence true. A symbol may be used more than once. Solve this problem on your paper.

Let the students work independently; then check the work collectively.

T (pointing to the blank box to the left): Which symbol could go here?

S: Minus.

Fill in the box with "-".

$$\boxed{7}\ \boxed{-}\ \boxed{2.5}\ \boxed{}\ \boxed{4}\ \boxed{=}\ \boxed{18}$$

T: What is 7 - 2.5 ?

S: 4.5.

T (pointing to the blank box to the right): Which symbol could go here?

S: Times, because $4.5 \times 4 = 18$.

---

[*] Hand-calculator sentences represent calculations that could be made with calculators that do chain calculations, where the boxes correspond to keys that would be pressed. In hand-calculator sentences, we assume all operations are performed in order from left to right; for example, "$\boxed{7}\ \boxed{+}\ \boxed{3}\ \boxed{\times}\ \boxed{2}$" represents the calculation "$(7 + 3) \times 2$" and "$\boxed{5}\ \boxed{\times}\ \boxed{3}\ \boxed{-}\ \boxed{1}\ \boxed{\div}\ \boxed{7}$" represents "$[(5 \times 3) - 1] \div 7$". See Section 3.1 for more information about hand-calculator sentences.

32 N

Repeat the preceding activity using the hand-calculator sentences suggested below.

$$\boxed{3.2}\ \boxed{}\ \boxed{5}\ \boxed{}\ \boxed{20}\ \boxed{=}\ \boxed{0.8}$$

Solution: $\boxed{3.2}\ \boxed{\times}\ \boxed{5}\ \boxed{\div}\ \boxed{20}\ \boxed{=}\ \boxed{0.8}$

$$\boxed{7.5}\ \boxed{}\ \boxed{2}\ \boxed{}\ \boxed{10}\ \boxed{=}\ \boxed{37.5}$$

Solution: $\boxed{7.5}\ \boxed{\div}\ \boxed{2}\ \boxed{\times}\ \boxed{10}\ \boxed{=}\ \boxed{37.5}$

Worksheets N47, N48, and N49 are available.

Name ____________ **N 47 ***

Put one of the symbols: +, ×, −, ÷ in each blank box to make the hand-calculator sentence true.

12 − 7 × 8 = 40

3 × 7 + 19 = 40

18 ÷ 3 × 7 = 42

4 × 7 + 0 = 28

35 ÷ 5 × 8 = 56

750 ÷ 10 ÷ 1 = 75

Name ____________ **N 48 ****

Put one of the symbols: +, ×, −, ÷ in each blank box to make the hand-calculator sentence true. A symbol may be used twice in one sentence.

1.8 × 10 ÷ 9 = 2

10 ÷ 4 − 0.7 = 1.8

2.6 + 3.9 ÷ 2 = 3.25

5.1 − 2.3 × 10 = 28

4.4 × 5 × 5 = 110

5 × 6 ÷ 100 = 0.3

0.7 + 0.8 × 5 = 7.5

Name ____________ **N 49 *****

Put one of the symbols: +, ×, −, ÷ in each blank box to make the hand-calculator sentence true. A symbol may be used more than once in one sentence.

5 × 8 ÷ 20 ÷ 5 = 0.4

1.7 + 4.3 × 7 ÷ 4 = 10.5

3 × 6 − 2 ÷ 8 = 2

0.4 + 0.6 ÷ 10 − 0.1 = 0

0 × or ÷ 387 + 25 ÷ 50 = 0.5

2 − 1.2 × 50 + or − 0 = 40

8 × 8 − 8 ÷ 8 = 7

5 ÷ 5 + 5 ÷ 5 = 1.2

6

MULTIPLICATION

This chapter provides continued and extended experiences in multiplication with arrow diagrams as the primary vehicle. Generalized relations for multiplication, symbolized by " ■ x", where " ■ " can be any whole number, are used in interesting arrow pictures that stimulate mental arithmetic involving composition of multiplication functions. Properly designed arrow pictures illustrate, for example, that 8 × 18 = 18 × 8; and the notion of the "product" of two numbers is introduced using cords.

CAPSULE LESSON SUMMARIES

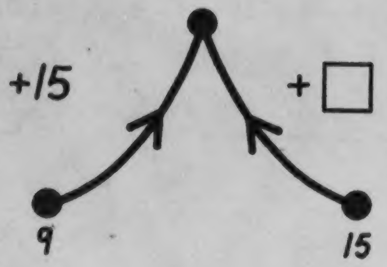
Section 6.1:	By labeling the dots and the arrows in specially designed arrow pictures, determine that, for example, 27 + 33 = 33 + 27 and 30 × 48 = 48 × 30. Introduce the meaning of the "product" of two numbers, and identify numbers at pairs of dots joined by red cords, where two numbers are joined by a red cord, for example, if and only if the product of the numbers is 36. (Worksheets N50, N51, N52, N53)
Section 6.2:	Label dots and identify arrows of the form " ■ x" in roads to and from a dot at a given number. Determine composites of relations in the arrow roads. (Worksheets N54, N55, N56, N57)

6 N

6.1 Multiplication #1

Exercise 1

Draw this arrow picture on the board.



T: What number is here (point to the unlabeled dot) and what could the blue arrow be for ?

S: The dot is for 24 and the arrow is for +9.

T: Do you notice anything interesting about this picture ?

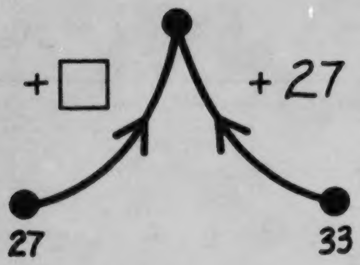
S: One dot is for 9 and the arrow opposite it is for + 9. The other dot is for 15 and the arrow opposite it is for + 15.

Accept any reasonable observation.

T: The arrow picture shows 9 + 15 = 15 + 9 = 24.

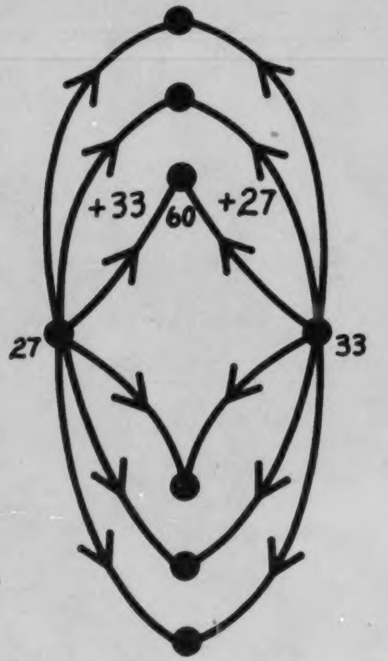
For emphasis, write the number sentence on the board next to the arrow picture.

Relabel the arrow picture as below.



Repeat the above activity using this arrow picture. The unlabeled dot is for 60 and the red arrow is for + 33.

Add these arrows to your picture.



6 N

Have the students copy the picture and label the dots and arrows. While the students are working independently, draw this table on the board.

$\xrightarrow{+\square}$	33							
$\xrightarrow{+\square}$	27							

After a few minutes, call the class' attention to the table and have some students complete the table. The table could look like this upon completion.

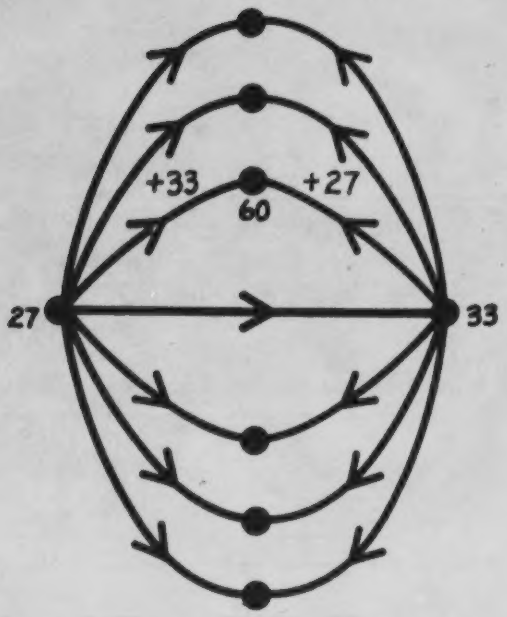
$\xrightarrow{+\square}$	33	8	10	15	30	37	25	100
$\xrightarrow{+\square}$	27	2	4	9	24	31	19	94

T: Do you see any patterns in our table ?

S: The difference between two red arrows is the same as the difference between their corresponding blue arrows.

S: Each red arrow is for plus six more than its blue arrow (e.g., 10 = 4 + 6).

Draw an arrow starting at 27 and ending at 33 in a different color.

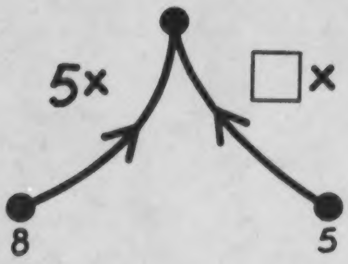

+33
+27
60
27
33

T: What could this arrow be for ?

S: +6.

Exercise 2

Draw this arrow picture on the board.



6 N

T: What number is here (point to the unlabeled dot) and what could the blue arrow be for? Write your answers on your paper.

S: The dot is for 40 and the blue arrow is for 8 ×.

Label the dot and the arrow.

T: What do you notice about this picture?

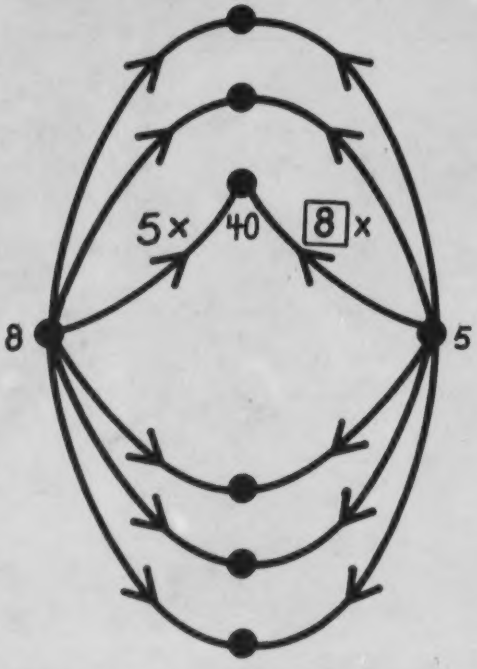
S: On the left "5 x 8" and on the right "8 x 5"; they are the same.

Accept any reasonable observation.

T: The arrow picture shows 8 × 5 = 5 x 8 = 40.

For emphasis, write the number sentence on the board next to the arrow picture.

Add these arrows to the picture.



Have the students copy the picture and label the dots and arrows. While the students are working independently, draw this table on the board.

□× →	5							
□× →	8							

After a few minutes, call the class' attention to your table and have some students complete the table. As the table is being completed, discuss the situation. The discussion might develop along these lines.

T: How did you label this red arrow (point to some red arrow) ?

S: 15 x .

T: If 5 × 8 = 40 (point to "40"), what is 15 × 8? (Point to the dot where a 15 x red arrow ends.)

S: 120.

Label the dot "120".

T: How did you calculate so quickly ?

S: 15 = 3 x 5 and 3 x 40 = 120.

T: How did you label the corresponding blue arrow ?

S: 24 x , because 3 x 8 = 24.

T: How did you label these two arrows ? (Point to another pair of red and blue arrows ending at the same point.)

6 N

S: 30x and 48x; I doubled 15 and 24.

Label a red arrow "30 x" and its corresponding blue arrow "48 x".

T: What is 30 x 8? (Point to the dot where the two arrows under discussion end.)

S: 240, the double of 120.

Label the dot "240".

Continue in this manner until you have a completed table.

Your table could look like this upon completion.

□x →	5	15	30	25	50	100	200	45
□x →	8	24	48	40	80	160	320	72

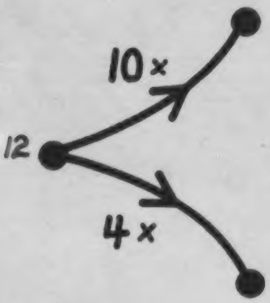
T: Do you see any patterns in our table?

S: The red arrows are for times a multiple of 5 and the blue arrows are for times (the same or corresponding) multiples of 8.

Accept any reasonable observations.

Exercise 3

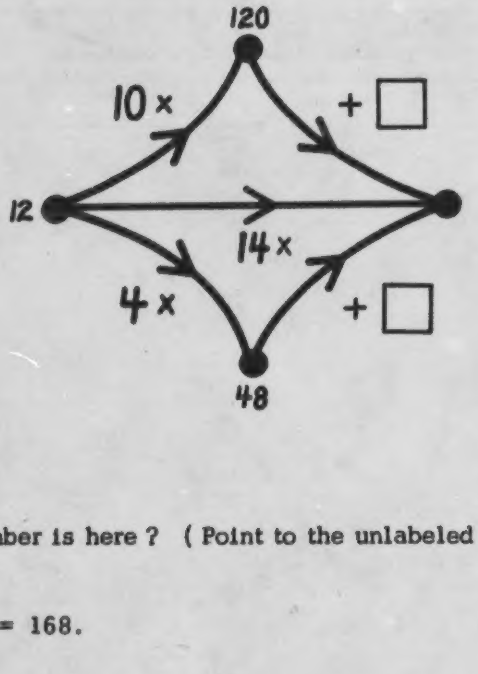
Draw this arrow picture on the board.



T: Label the dots.

S: 10 x 12 = 120 and 4 x 12 = 48.

Add these arrows to your picture.



T: What number is here? (Point to the unlabeled dot.)

S: 14 x 12 = 168.

T: Now, how should we label the two arrows?

6 N

S: +48 and +120.

T: What do you notice about the picture?

S: 120 + 48 = 48 + 120 = 168.

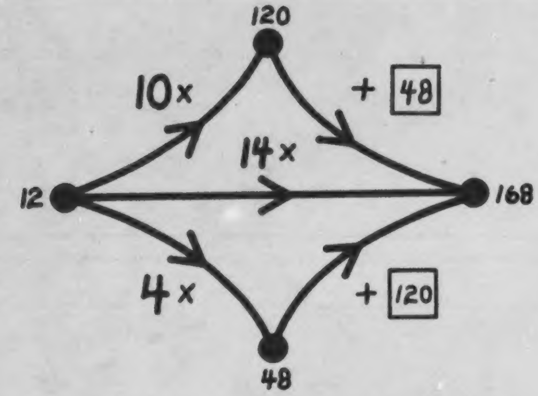
T: The picture also show that

$$14 \times 12 = (10 \times 12) + (4 \times 12)$$

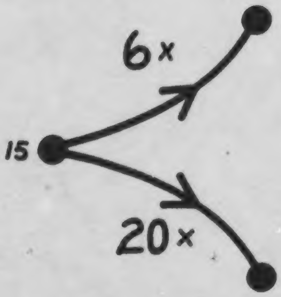
and

$$14 \times 12 = (4 \times 12) + (10 \times 12).$$

A completed arrow picture is drawn below.

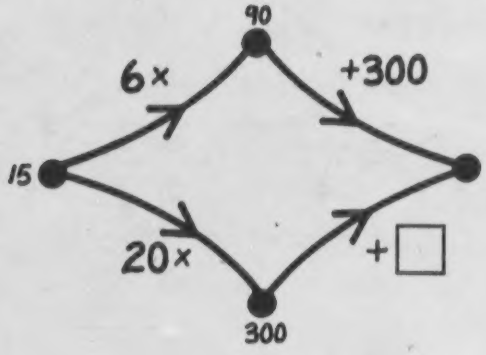

120
10x
+ 48
14x
12
168
4x
+ 120
48

Draw this arrow picture and have the students label the dots.

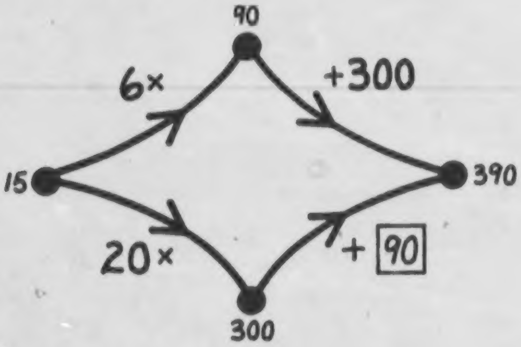


246

Next, add these arrows to your picture. Ask the students to copy the picture and complete it.


90
6x
+300
15
20x
+☐
300

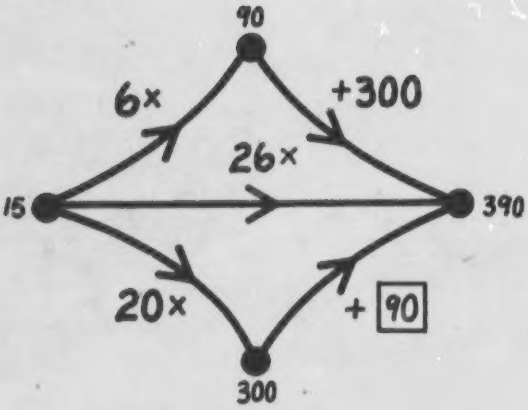
After a few minutes, have students complete the picture on the board.


90
6x
+300
15
390
20x
+ 90
300

Draw a blue arrow starting at 15 and ending at 390.

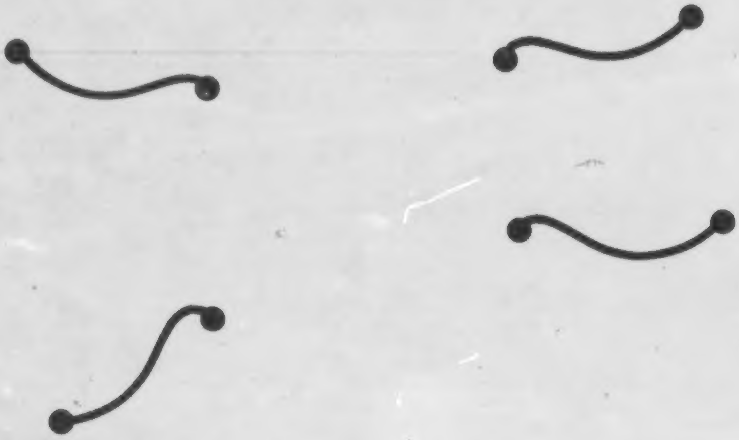
T:	What could this blue arrow be for ?
S:	26x; because 26 times a number is the same as 6 times that number plus 20 times the same number.

Label the blue arrow "26 x".



Exercise 4

Draw this picture on the board.



T: Two numbers are joined by a red cord if and only if their product equals 24. What is the product of 6 and 4 ?

S: 24, because 6 x 4 = 24.

T: That's correct. The product of two numbers is the number you get when you multiply them together. Which numbers could be joined by a red cord ?

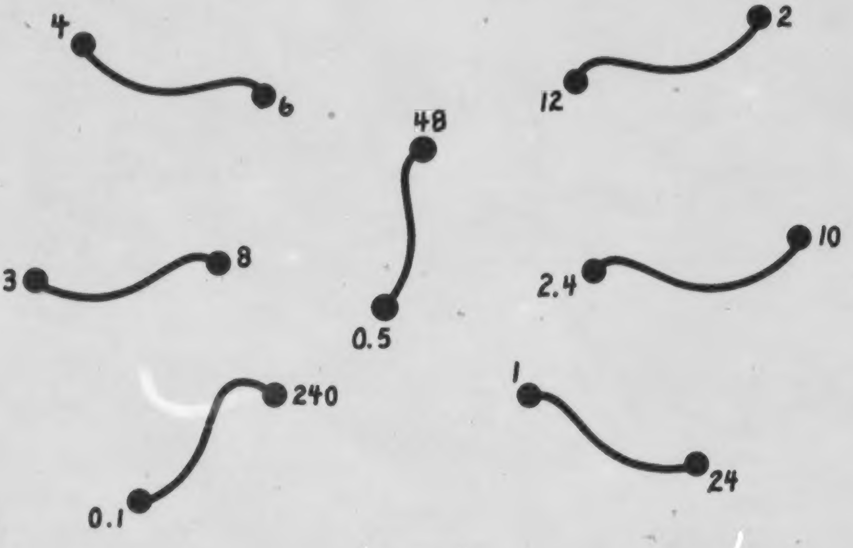
S: 6 and 4, because 6 × 4 = 24.

Label a pair of dots joined by a cord "6" and "4".

S: 48 and 0.5, because 48 × 0.5 = 24.

Label another pair of dots "48" and "0.5". When it is obvious that most students understand the rule for the cords, ask the students to draw more dots and red connecting cords on their paper and to label the dots.

Let the students work independently for several minutes. Extend your picture and have students label the dots.[*] A partially completed picture is shown below.



Worksheets N50, N51, N52, and N53 are available.

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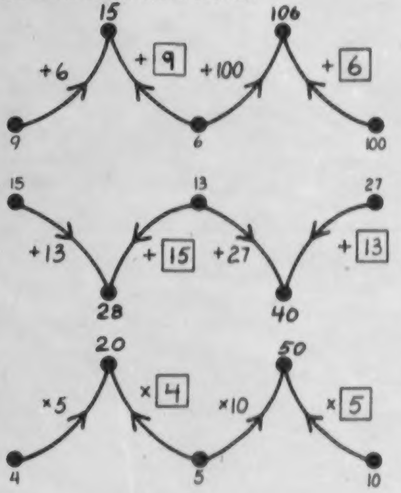
[*] Although it would be correct for students to use negative numbers to label both dots in a pair (for example, $^{-}1$ and $^{-}24$, or $^{-}3$ and $^{-}8$, or $^{-}10$ and $^{-}2.4$, etc.), at this point in the curriculum, there is no way for you to explain to them why it is correct. We suggest you encourage students, if necessary, to use only positive numbers.

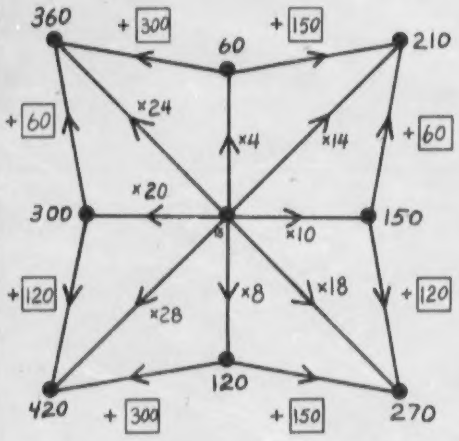
Name ____________________ N 50 *

Label the dots and fill in the boxes for the arrows.


15
106
+6
+ 9
+100
+ 6
9
6
100
15
13
27
+13
+ 15
+27
+ 13
28
40
20
50
×5
× 4
×10
× 5
4
5
10

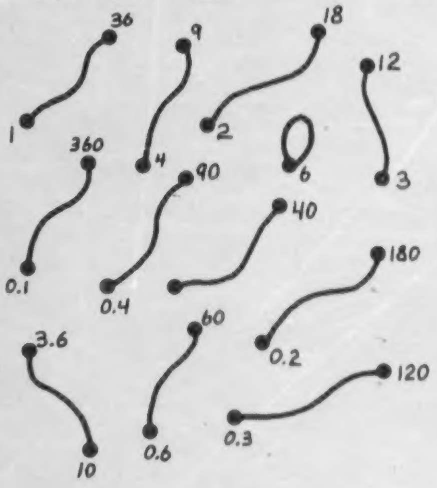
Name ____________________ N 51 **

Label the dots and fill in the boxes for the arrows.


360
+ 300
60
+ 150
210
×24
+ 60
×4
×14
+ 60
×20
300
6
×10
150
+ 120
×8
×18
+ 120
×28
120
420
+ 300
+ 150
270

Name ____________________ N 52 ***

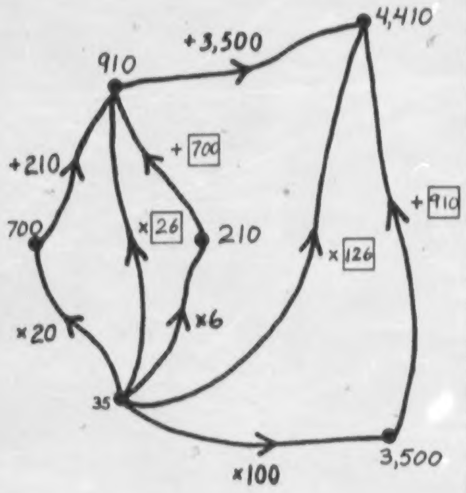
Two numbers are joined by a red cord if and only if their product equals 36.
Label the dots. Many answers are possible.


36
9
18
12
1
2
4
6
3
360
90
40
0.1
0.4
180
3.6
60
0.2
120
10
0.6
0.3

(Many other solutions are possible.)

Name ____________________ N 53 ****

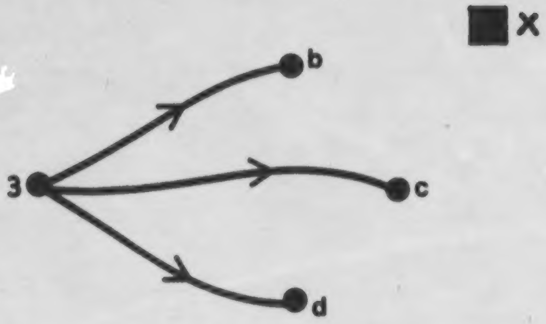
Label the dots and fill in the boxes for the arrows.


4,410
+3,500
910
+ 700
+210
+ 910
700
× 26
210
× 126
×6
×20
35
3,500
×100

6.2 Multiplication #2

Exercise 1

Draw this arrow picture on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: The red arrows are for multiplication by some whole number.

Point to "3" and trace a red arrow ending at b.

T: Some whole number times 3 equals this number (at b). Could this number be 6?

S: Yes, if the red arrow were for 2 x.

T: Could the number (at b) be 15?

S: Yes, if the red arrow were for 5 x.

T: Could the number be 1.5?

S: Yes, if the red arrow were for $\frac{1}{2}$ x.

S: But $\frac{1}{2}$ is not a whole number.

T: So this number (at b) cannot be 1.5.
What other number could be here (at b)?

The class discussion might develop along these lines.

S: 30.

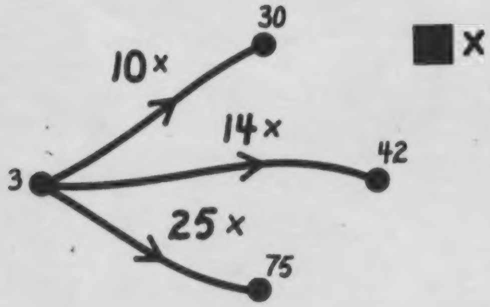
Label b "30".

T: In order for "30" to go here, we have to find a whole number so that the red arrow is for times that whole number. What whole number will "work"?

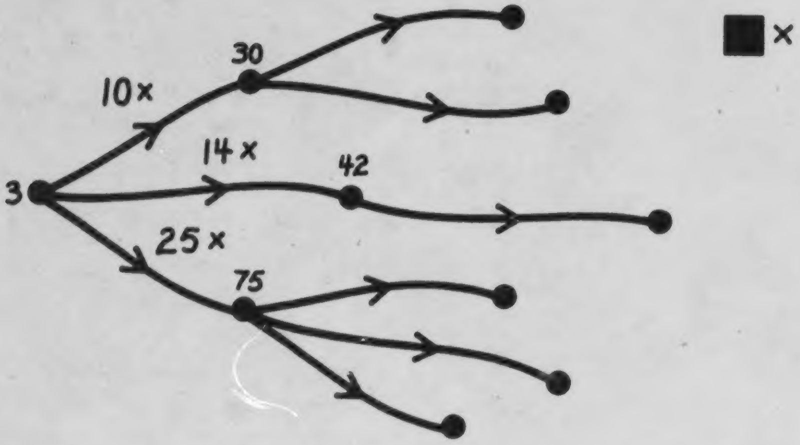
S: 10.

Label the arrow (starting at 3 and ending at 30) "10 x ".

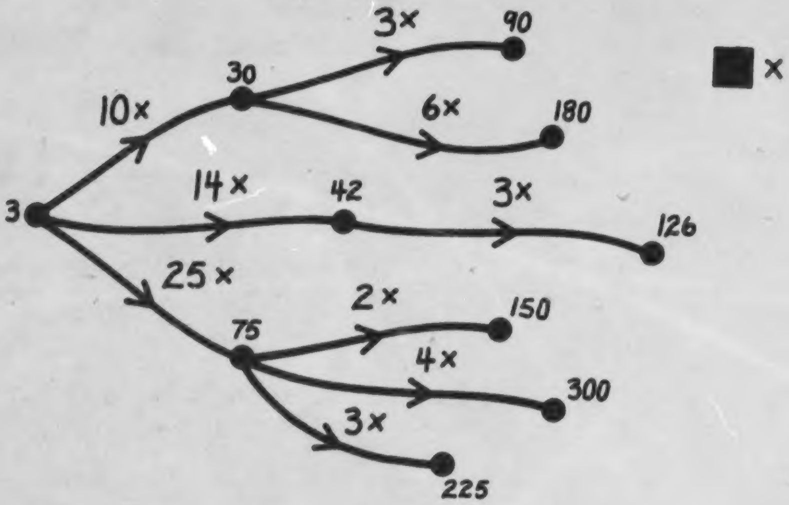
Continue in this manner for c and d, each time labeling a dot, then the corresponding arrow. If you think it would be beneficial for the students, erase the numerals and repeat the activity. A completely labeled arrow picture might look like this.



Extend your arrow picture with the addition of these arrows and continue to label dots and arrows.

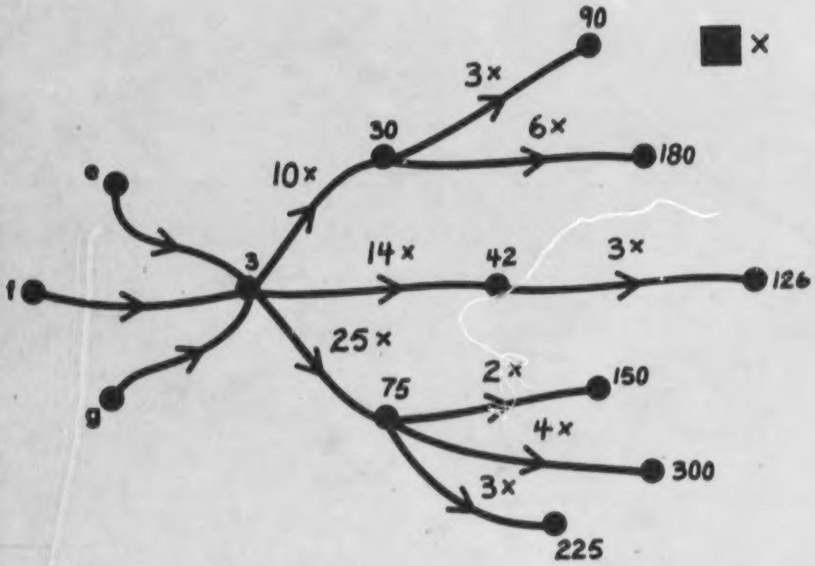

3
10×
30
14×
42
25×
75
■ ×

Your arrow picture might look like this upon completion of this activity.


3×
90
30
10×
6×
180
14×
42
3×
3
126
25×
2×
150
75
4×
300
3×
225
■ ×

15 N

Add these arrows ending at 3 to your picture.

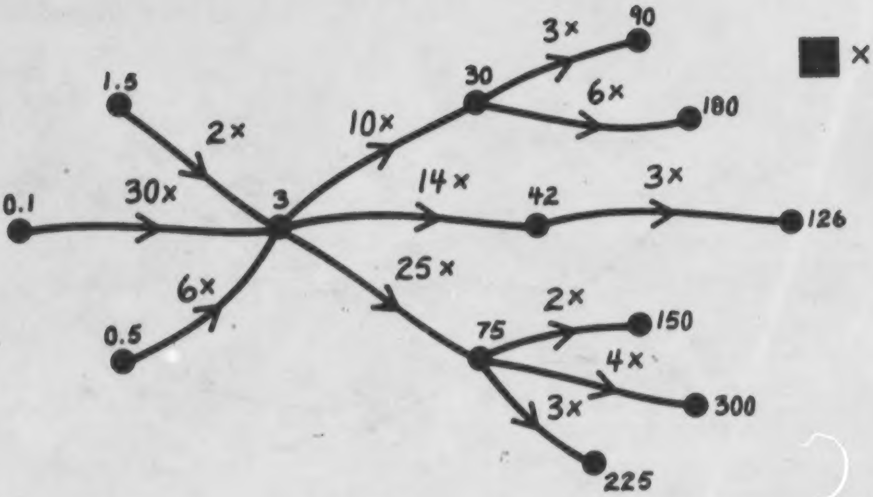


Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: Copy the three arrows ending at 3 on your paper. Label the dots as you wish, but be sure each red arrow can be for multiplication by some whole number. The dots may be for (non-integer) decimal numbers.

Let students work independently on this problem for a few minutes. Ask faster workers to add more arrows ending at 3. Before concluding this activity, have the students label e, f, and g and the respective arrows at the board.

Your arrow picture might look like the one on the following page.


■ ×
0.1
30x
1.5
2×
0.5
6×
3
10×
30
3×
90
6×
180
14×
42
3×
126
25×
75
2×
150
4×
300
3×
225

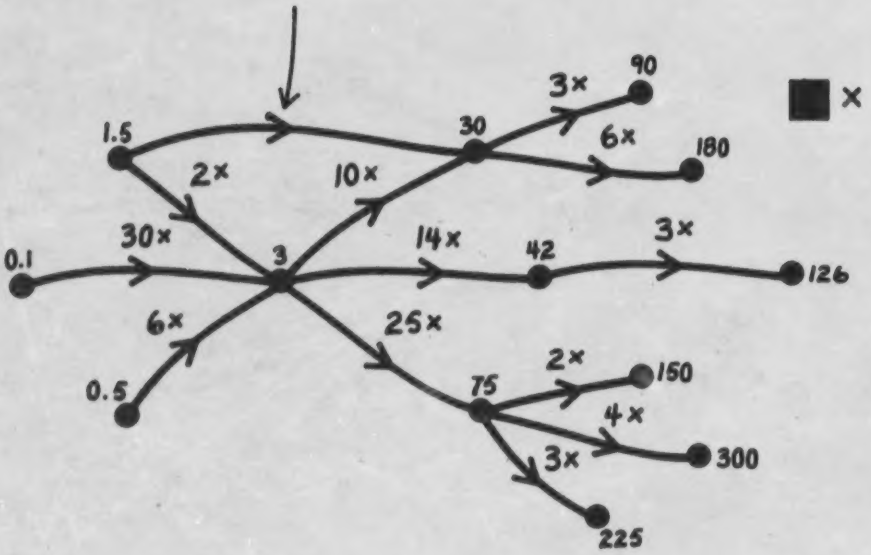
T (pointing to "30", "42", and "75"): What do we call numbers like these?

S: Multiples of 3.

T: Very good. Do you see any other red arrows that we could draw?

Have a student trace an arrow; then you draw it. Suppose an arrow starting at 1.5 and ending at 30 is drawn (see the next illustration).

15 N


■ ×
90
3×
1.5
30
6×
180
2×
10×
30×
14×
42
3×
0.1
3
126
6×
25×
2×
0.5
75
150
4×
3×
300
225

T: This arrow must be for times some whole number. We can use the picture to find this whole number.

Start at 1.5 and trace the 2 × arrow followed by the 10 × arrow, saying:

T: 2 × followed by 10 ×.

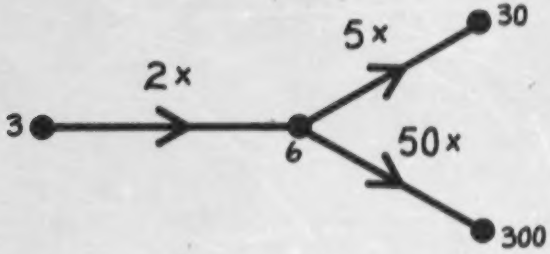
S: 20 ×.

Ask for a few more red arrows that can be drawn and label them times some whole number. Each time a red arrow is added, trace the appropriate arrows to stress that a label for the arrow can be found from a composition of two arrows.

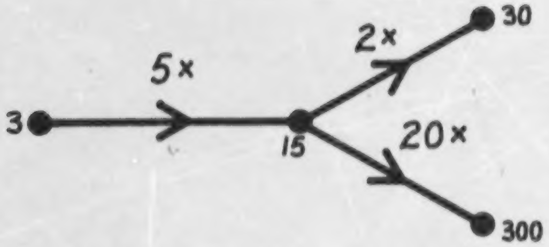
There are many possibilities; a few are shown on the following page.

T: Copy this picture on your paper. Label the dot and the arrows. Remember, each arrow has to be labeled times some whole number.

There are two solutions to this problem shown below. After the students have had a few minutes to find a solution, have both solutions put on the board.



■ ×

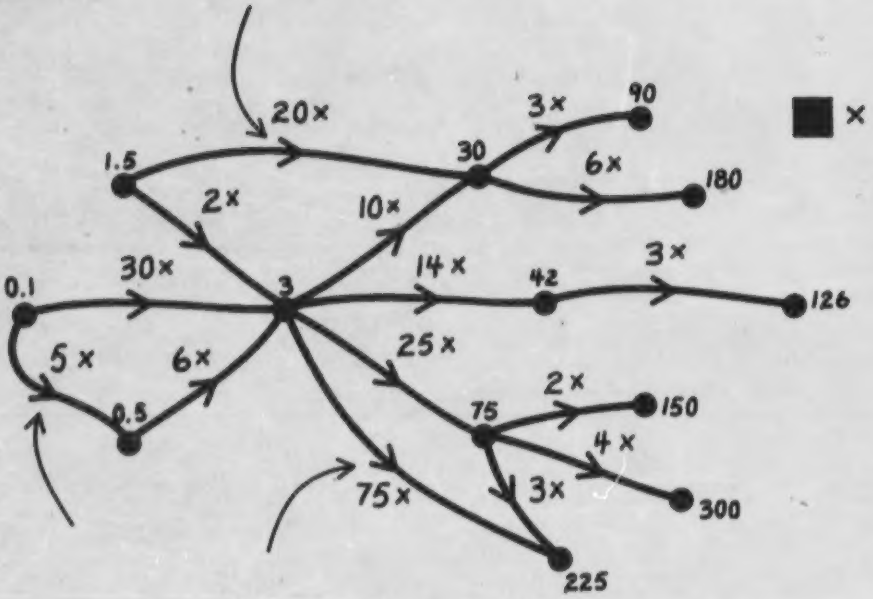


Point to either of the arrow pictures and ask:

T: Can you see where more red arrows can be drawn in our picture?

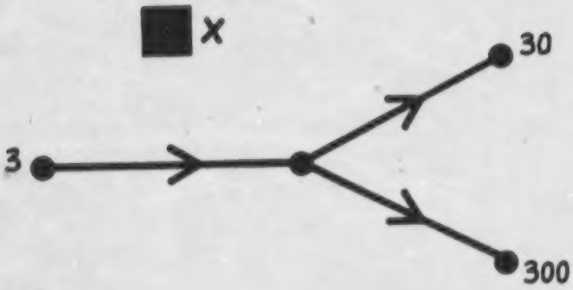
Have students draw and label red arrows. The next illustrations show completed arrow pictures.

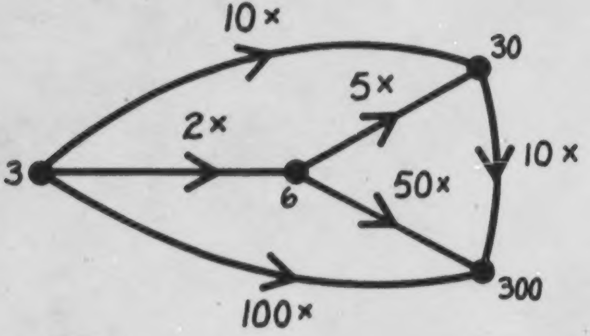
15 N

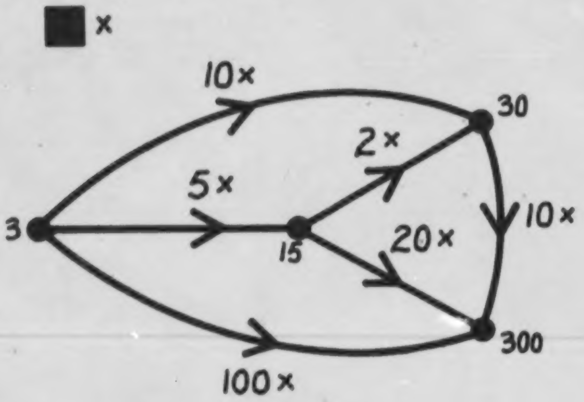

■ ×
20×
3×
90
1.5
30
6×
180
2×
10×
0.1
30×
3
14×
42
3×
126
5×
6×
25×
0.5
75
2×
150
4×
3×
300
75×
225

Exercise 2

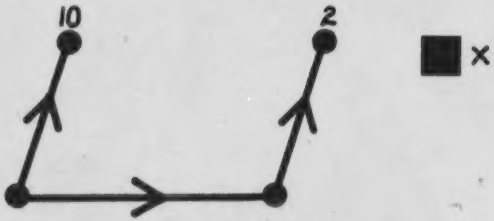
Draw this arrow picture on the board.




10 ×
30
5 ×
2 ×
10 ×
3
6
50 ×
300
100 ×


■ ×
10 ×
30
2 ×
5 ×
10 ×
3
15
20 ×
300
100 ×

Draw this arrow picture on the board.

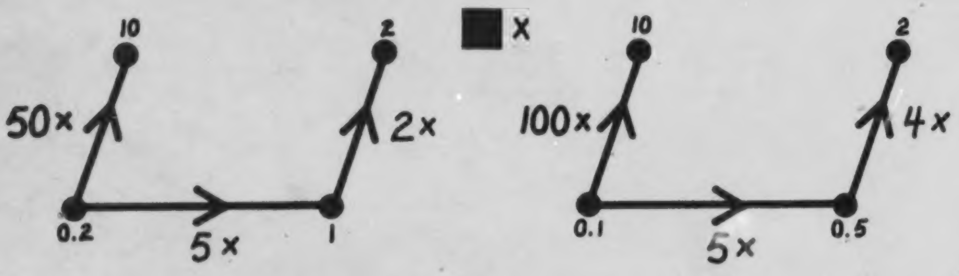


T: Copy this picture and label the dots and arrows. Remember, for each red arrow, find some whole number so that the arrow is for times that whole number. There are many possible solutions.

Let the students work on this problem independently for a few minutes. If time permits, put several solutions on the board. Otherwise, put at least one solution on the board.

15 N

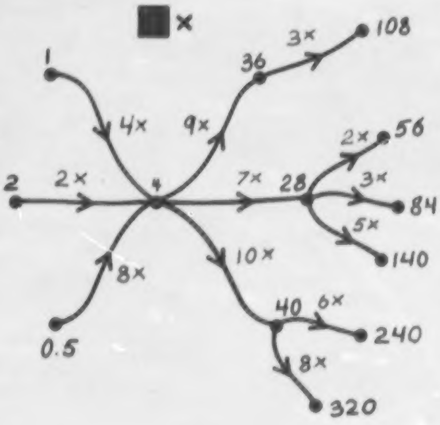
Two of the possible solutions are shown below.


■ x
10
2
50×
2×
0.2
5×
1
10
2
100×
4×
0.1
5×
0.5

Worksheets N54, N55, N56, and N57 are available.

Name ______________ **N 54** *

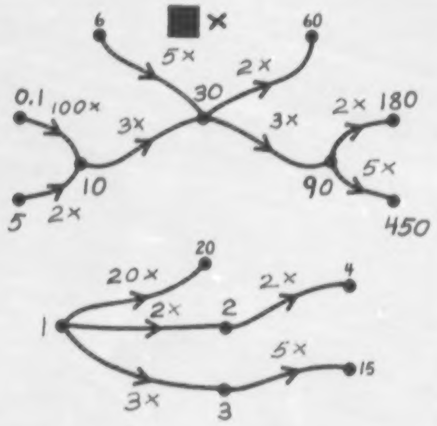
" ■ × " means each red arrow is for times some WHOLE number.
Label the dots. Label each arrow " ☐ × ". Many solutions are possible.


■ ×
1
4×
2
2×
4
9×
36
3×
108
7×
28
2× 56
3× 84
5× 140
10×
40
6× 240
8× 320
8×
0.5

(Many other solutions are possible.)

Name ______________ **N 55** **

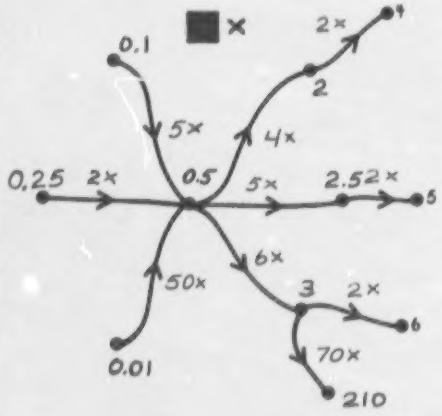
" ■ × " means each red arrow is for times some WHOLE number.
Label the dots. Label each arrow " ☐ × ". Many solutions are possible.


6
■ ×
60
5×
2×
30
0.1
100×
3×
3×
2× 180
10
90
5×
5
2×
450
20
20×
2×
4
2×
2
1
5×
15
3×
3

(Many other solutions are possible.)

Name ______________ **N 56** ***

" ■ × " means each red arrow is for times some WHOLE number.
Label the dots. Label each arrow " ☐ × ". Many solutions are possible.

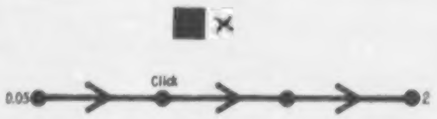

■ ×
2×
4
0.1
2
5×
4×
0.25
2×
0.5
5×
2.5
2×
5
6×
50×
3
2×
6
70×
0.01
210

(Many other solutions are possible.)

Name ______________ **N 57** ****

First clue

" ■ × " means each red arrow is for times some WHOLE number.



Click is 0.1 or 0.2 or 0.25 or 0.5.

Second clue

If you multiply Click by 20, you will get a square number.

Click is 0.2.

7

RATIONAL NUMBERS

Using a variety of situations, stories and games which involve extensive use of arrow pictures, probability and special abaci, the lessons in this chapter reinforce and extend the properties of rational numbers and rational multiplication functions. Those properties being reinforced are illustrated by:

a) $\frac{1}{3}\times$ is the same function as $\div 3$ and is the converse of $3\times$.

b) $\frac{2}{3}\times$ is the composite of $2\times$ followed by $\div 3$ or of $\div 3$ followed by $2\times$.

c) a rational number may have many fractional names.

These properties are extended to:

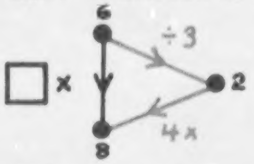
d) a rational number may have many decimal names.

e) the composite of $\frac{4}{5}\times$ and $\frac{2}{3}\times$ is $\frac{8}{15}\times$, and $\frac{4}{5}\times\frac{2}{3}=\frac{8}{15}$.

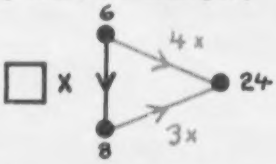
In a probability setting, the order of rational numbers is introduced: for example, 4 chances in 5 is better than 2 chances in 3 so that $\frac{4}{5}>\frac{2}{3}$.

CAPSULE LESSON SUMMARIES

Section 7.1: Label each arrow in a given picture as "$\square\times$", where "$\square$" is replaced by a whole number or a fraction. Make use of detours in doing this; for example,



or



(Worksheets N58, N59, N60, N61)

Section 7.2: By imagining results of spinning spinners or of drawing marbles from a box, determine which of two rational numbers is the larger. Do this by comparing certain equivalent fractions for each of the numbers. Estimate which is larger, $\frac{3}{5}$ of a number or $\frac{2}{3}$ of the number, $\frac{2}{3}$ of a number or $\frac{3}{2}$ of the number, etc.
(Worksheets N62, N63, N64, N65)

Section 7.3: Extend the binary abacus to boards on the right of the green bar, and use plays which follow the binary abacus rule to help label these boards. Put numbers on this abacus and decode numbers. Use this abacus to determine that $\frac{1}{3} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots$. By sharing a cake equally among four friends, and then again sharing one of the portions equally, and then again, . . . , verify that $\frac{1}{3}$ of the cake is $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots$ of the cake.

Section 7.4: Through a story of a system of dangerous roads, with probabilities given of surviving various of the roads, determine the probability of surviving a trip along a certain path through the system. Find that, for example, $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12} = \frac{1}{2}$.

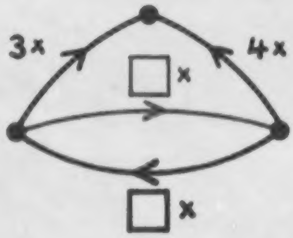
Section 7.5: Play Guess My Rule games in which the rule is given by an arrow; first play the game with arrows for the function $\frac{2}{3}x$, then with arrows for "2 times the starting number plus the ending number is 20".

(Worksheets N68, N69, N70)

Section 7.6: Extend the Base Three abacus (three checkers on a board represents the same number as one checker on the next board to the left) to boards right of the green bar, and label these boards by making plays which follow the rule of this abacus. Using this abacus, determine that $\frac{1}{2} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots$. By sharing a cake equally among three friends, and then again . . . , verify that $\frac{1}{2}$ of the cake is the same as $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$ of the cake.

(Worksheet N71)

Section 7.7: Using a composition of multiplication and division functions, label arrows in pictures such as:

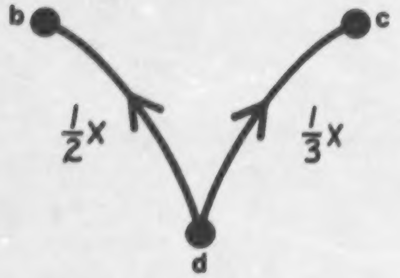


(Worksheets N72, N73, N74, N75)

7.1 Fractions #1

Exercise 1

Draw this arrow picture on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: Each dot in this picture is for a whole number. What number could be here (point to d)?

S: 6.

T: If this number is 6 (point to d), what is this number (point to b)?

S: 3, because $\frac{1}{2} \times 6 = 3$.

T: And what is this number (point to c)?

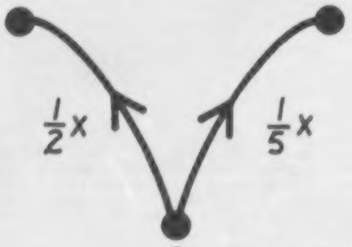
S: 2, because $\frac{1}{3} \times 6 = 2$.

Let students suggest several numbers that could be at d. Check each suggestion to be sure that b and c are whole numbers. Encourage many students to participate in the checks as this provides practice with $\frac{1}{2} \times$ and $\frac{1}{3} \times$. As numbers for d are suggested, make a list of these numbers on the board. Your list may include these numbers: 6; 12; 18; 24; 30; 36; etc.

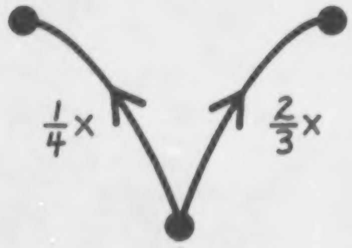
T: What do we call numbers like these in our list?

S: Multiples of 6.

Repeat the activity above using these arrow pictures.



(e can be any multiple of 10)



(f can be any multiple of 12)

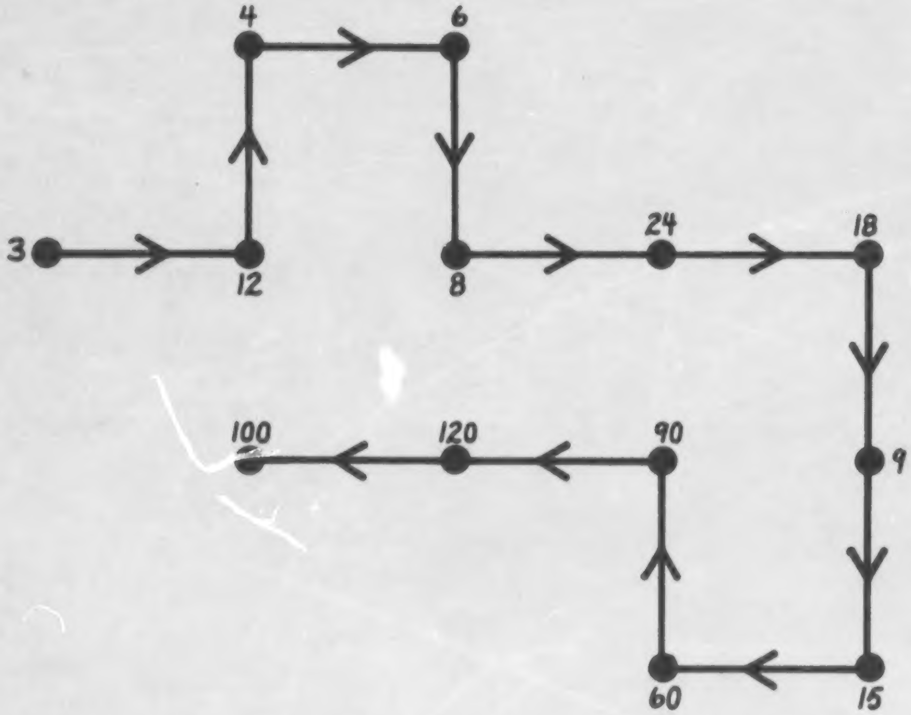
It might be necessary to remind students of the meaning of "$\frac{2}{3}$ x". Do this while checking some students' suggestions. Suppose a student suggests f could be 12.

T: How do you calculate "$\frac{2}{3} \times 12$"?

S: Divide 12 by 3 and multiply the result by 2 (or multiply 12 by 2 and divide the product by 3). So, $\frac{2}{3} \times 12 = 8$.

Exercise 2

Slowly and progressively draw this arrow road on the board.



T: Label each arrow "☐ x".

The first two arrows should be labeled "4 x" and "$\frac{1}{3}$ x" respectively. Labeling the third arrow is more challenging.

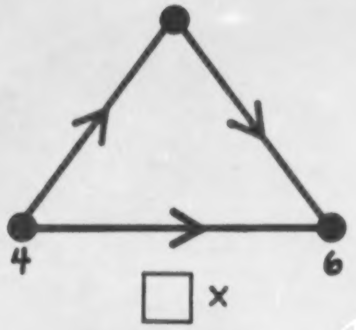
T (tracing the arrow from 4 to 6): Label this arrow "☐ x".

S: May we take a detour?

T: Yes, that's a good idea.

8N

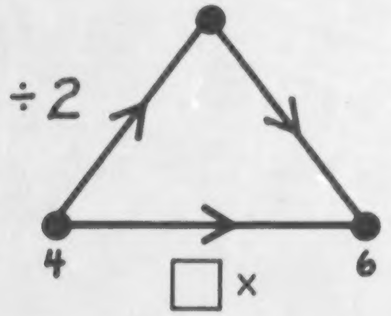
If no one suggests this idea, do so yourself. Add these arrows to your drawing.



T: We would like to label one of these arrows "☐ x" and the other "÷ ☐ ". What could the arrows be for?

S: One arrow could be for ÷ 2.

Label the arrow to the left "÷ 2".



T: What number is 4 ÷ 2?

S: 2.

Label the dot "2".

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T: What could the second blue arrow be for?

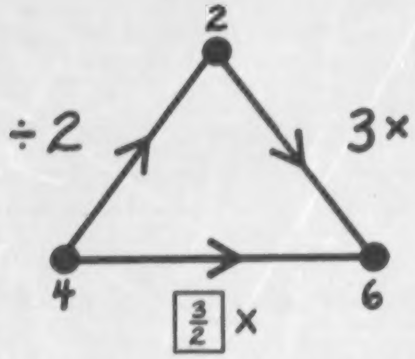
S: 3x.

Label the arrow "3x".

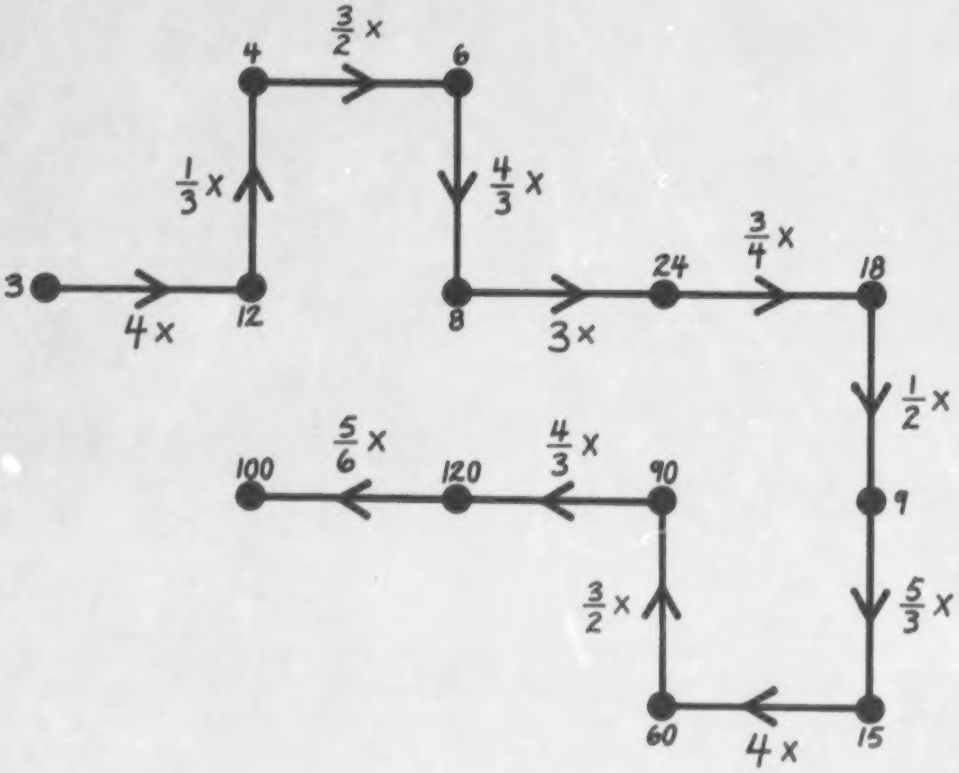
T (tracing the blue arrows): ÷2 followed by 3x. What could the red arrow be for?

S: $\frac{3}{2}$x.

Label the red arrow "$\frac{3}{2}$x" and continue the activity

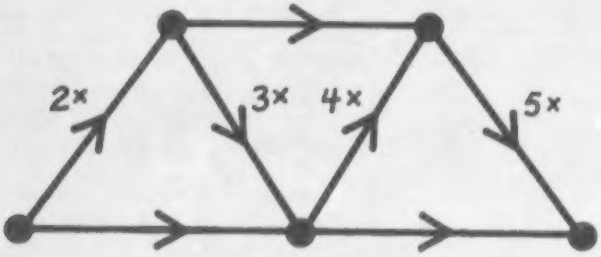


A completed arrow road is shown in the next illustration. It may happen that equivalent names other than those shown below will be suggested by some students. For example, some equivalent names for the function $\frac{3}{2}$x are: "$\frac{6}{4}$x"; "$\frac{9}{6}$x"; "$\frac{12}{8}$x"; and so on. Accept any correct answer.


3
4x
12
1/3 x
4
3/2 x
6
4/3 x
8
3x
24
3/4 x
18
1/2 x
9
5/3 x
15
4x
60
3/2 x
90
4/3 x
120
5/6 x
100

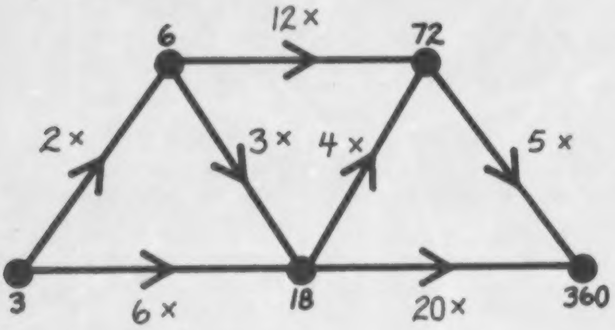
Exercise 3

Draw this arrow picture on the board and ask the students to copy it on their paper.

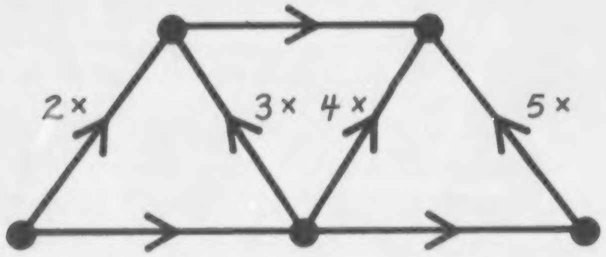


T: Label each blue arrow. They may have different labels. Then label the dots to check your answer.

Let the students work independently for a few minutes. Then check the work collectively. A completed arrow picture with the dots labeled as students suggested is shown below.



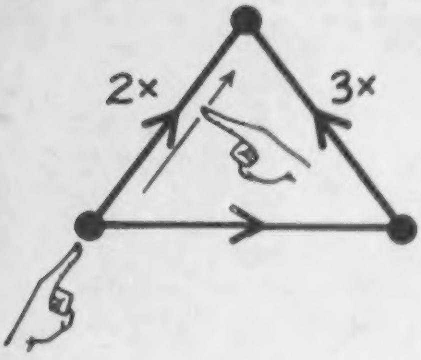
Repeat the above activity using this arrow picture.

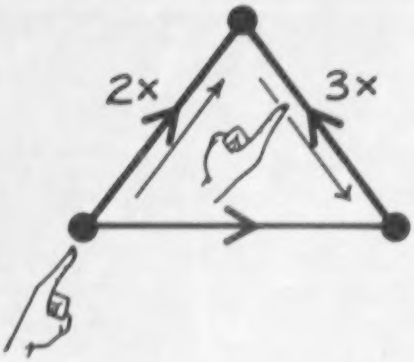


T: Look carefully at the direction of the arrows. How could we label each blue arrow? They may have different labels.

Let the students work independently for a few minutes. If students have difficulty getting started, trace with the 2 x arrow and against the 3 x arrow as shown in the next illustration.

8 N





T (with the 2 x arrow followed by against the 3 x arrow): What is another name for "against 3 x"?

S: "÷ 3".

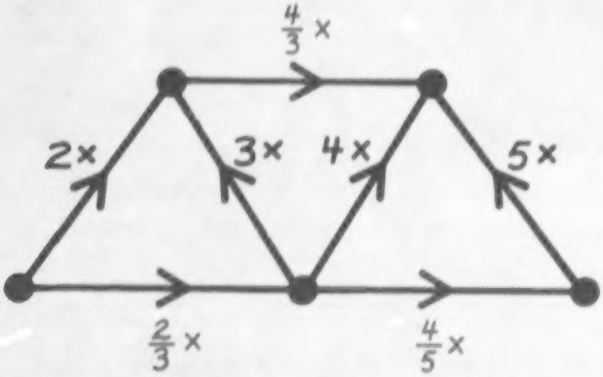
T: 2 x followed by ÷ 3 is the same as ... ?

S: $\frac{2}{3}$ x.

Label the blue arrow resulting from the composition of 2 x and ÷ 3, "$\frac{2}{3}$x".

Let the students work independently to label the remaining arrows. Then check the work collectively.

A completed arrow picture is shown below.

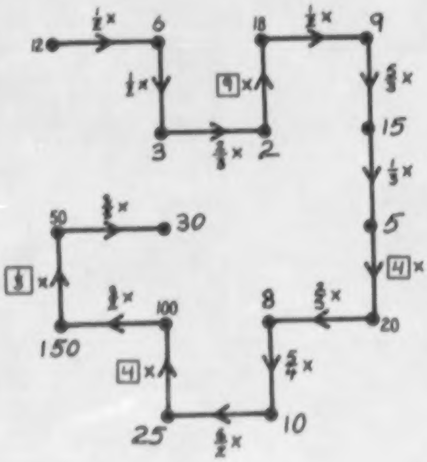


Worksheets N58, N59, N60, and N61 are available.

Name ______________________

N 58 *

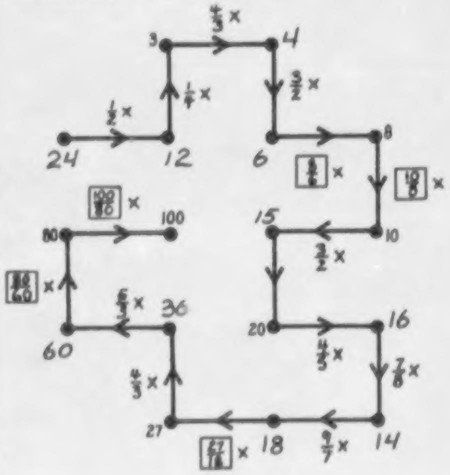
Label the dots and fill in the boxes for the arrows.


12
1/2 ×
6
1/2 ×
3
2/3 ×
2
9 ×
18
1/2 ×
9
5/3 ×
15
1/3 ×
5
4 ×
20
2/5 ×
8
5/4 ×
10
5/2 ×
25
4 ×
100
3/2 ×
150
1/3 ×
50
3/5 ×
30

Name ______________________

N 59 **

Label the dots and fill in the boxes for the arrows.

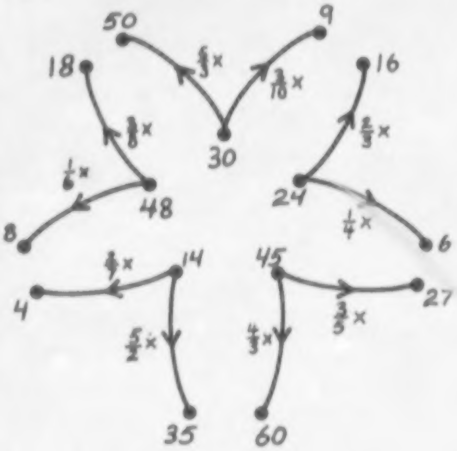

24
1/2 ×
12
1/4 ×
3
4/3 ×
4
3/2 ×
6
8/6 ×
8
10/8 ×
10
3/2 ×
15
20
4/5 ×
16
7/8 ×
14
9/7 ×
18
27/18 ×
27
4/3 ×
36
5/3 ×
60
80/60 ×
80
100/80 ×
100

(The answers above are those that may be given by students. It is possible that other correct names could be given instead, e.g., $\frac{8}{6} = \frac{4}{3} = \frac{12}{9} = \frac{16}{12} = \ldots$)

Name ______________________

N 60 ***

All the dots are for positive WHOLE numbers.
Label the dots. Many answers are possible.


30
5/3 ×
50
3/10 ×
9
48
3/8 ×
18
1/6 ×
8
24
2/3 ×
16
1/4 ×
6
14
2/7 ×
4
5/2 ×
35
45
3/5 ×
27
4/3 ×
60

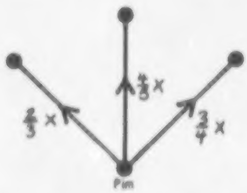
(Many solutions are possible.)

Name ______________________

N 61 ****

First clue

In this picture, all the dots are for positive WHOLE numbers.



Pim could be 60 ; 120 ; 180 ; 240 ; 300 ; 360 ; 420 ; 480 ; 540 ; 600 ; and so on.

Second clue

Pim is a square number smaller than 1,000.

Pim is 900 .

11 N

7.2 Fractions #2

Exercise 1

Draw this spinner on the board.

T: If I spin this spinner and it stops on red, I win. What is the probability of winning?

S: $\frac{2}{5}$ (or 2 out of 5).

T: Suppose I roll a die. If a divisor of 6 comes up, I win. What is the probability of winning?

S: 4 chances out of 6.

S: $\frac{4}{6} = \frac{2}{3}$.

T: Suppose I shake 2 red and 6 blue marbles, then put them behind my back and choose one. If I choose a blue one, I win. What is the probability of winning?

S: $\frac{6}{8}$ or $\frac{3}{4}$.

Exercise 2

Write this information on the board.

FIRST	SECOND	THIRD
$\frac{3}{4}$	$\frac{2}{3}$	$\frac{6}{8}$

T: Bruce was playing a game and he could choose one of three random devices (spinners, coins, dice, marbles, etc.) to play with. With the first device he had a probability of $\frac{3}{4}$ of winning, with the second device a probability of $\frac{2}{3}$, and with the third device a probability of $\frac{6}{8}$.

On your paper, write the probability of the device you think Bruce should choose. Remember, Bruce likes to win so he wants the largest probability of winning possible.

While the students are solving this problem, walk around and observe their choices.

T: Many of you chose either $\frac{3}{4}$ or $\frac{6}{8}$. Which one is better?

Let the students lead the discussion of this situation and those that follow.

S: They are the same; $\frac{3}{4} = \frac{6}{8}$.

T: Can you give me some other names for $\frac{3}{4}$?

Accept any equivalent name for $\frac{3}{4}$, e.g., "$\frac{6}{8}$", "$\frac{9}{12}$", "$\frac{12}{16}$", "$\frac{15}{20}$", ... List whatever correct names are suggested in a string of equalities:

$$\frac{3}{4} = \frac{6}{8} = \frac{9}{12} = \frac{12}{16} = \frac{15}{20} = \ldots$$

T: Why wouldn't you choose $\frac{2}{3}$?

S: $\frac{2}{3}$ is smaller than $\frac{3}{4}$.

T: Can you explain how you know that?

Some students may know that $\frac{2}{3}$ is smaller than $\frac{3}{4}$ and not be able to explain why. For the benefit of these students, ask:

T: Can you give me some other names for $\frac{2}{3}$?

Accept any equivalent name for $\frac{2}{3}$, e.g., "$\frac{4}{6}$", "$\frac{6}{9}$", "$\frac{8}{12}$", "$\frac{10}{15}$", ... Be sure to list whatever correct names are suggested in a string of equalities:

$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12} = \frac{10}{15} = \ldots$$

S: $\frac{2}{3} = \frac{8}{12}$ and $\frac{3}{4} = \frac{9}{12}$, so $\frac{2}{3}$ is smaller than $\frac{3}{4}$.

S: $\frac{3}{4} = \frac{6}{8}$ and $\frac{2}{3} = \frac{6}{9}$ and $\frac{6}{9}$ is smaller than $\frac{6}{8}$.

Your students may prefer decimal names to fractions. For example,

S: $\frac{3}{4} = 0.75$ and $\frac{2}{3} = 0.666\ldots$, so $\frac{2}{3}$ is smaller than $\frac{3}{4}$.

Here is another valid argument that one of your students may give.

S: $\frac{2}{3}$ is smaller than $\frac{3}{4}$ because $\frac{2}{3} = 1 - \frac{1}{3}$ and $\frac{3}{4} = 1 - \frac{1}{4}$ and $\frac{1}{3}$ is larger than $\frac{1}{4}$.

Change the probabilities of the three random devices on the board.

FIRST	SECOND	THIRD
$\frac{3}{4}$	$\frac{4}{5}$	0.8

T: Bruce was playing another game where he could choose one of three random devices. This time the winning probabilities were: $\frac{3}{4}$; $\frac{4}{5}$; and 0.8. On your paper, write the probability of the device you think Bruce should choose.

While the students are solving this problem, walk around and observe their responses.

T: Many of you chose either $\frac{4}{5}$ or 0.8. Which one is better?

S: They are the same; $0.8 = \frac{8}{10} = \frac{4}{5}$.

T: Can you convince me that $\frac{4}{5}$ or 0.8 is larger than $\frac{3}{4}$?

Let the students have a minute or two to consider the problem.

S: $\frac{4}{5} = 0.8 = 0.80$ and $\frac{3}{4} = 0.75$, so $\frac{4}{5}$ is larger than $\frac{3}{4}$.

S: $\frac{4}{5} = \frac{16}{20}$ and $\frac{3}{4} = \frac{15}{20}$, so $\frac{4}{5}$ is larger than $\frac{3}{4}$.

S: $\frac{4}{5} = \frac{12}{15}$ and $\frac{3}{4} = \frac{12}{16}$, so $\frac{4}{5}$ is larger than $\frac{3}{4}$.

S: $\frac{4}{5} = 1 - \frac{1}{5}$ and $\frac{3}{4} = 1 - \frac{1}{4}$ but $\frac{1}{4}$ is larger than $\frac{1}{5}$, so $\frac{4}{5}$ is larger than $\frac{3}{4}$.

11 N

Exercise 3

T: You have a choice of $\frac{2}{7}$ of a cake or $\frac{1}{3}$ of a cake. Which piece will you choose? Why? Write your choice on your paper.

S: I chose $\frac{1}{3}$ because $\frac{1}{3} = \frac{2}{6}$ and $\frac{2}{6}$ is larger than $\frac{2}{7}$.

T: Suppose instead you have a choice between $\frac{3}{5}$ of a cake and $\frac{4}{7}$ of a cake. Which piece of cake would you choose? Write your choice on your paper.

Let the students have some time to consider this problem. Some students might suggest using equivalent fractions. In that case, invite the students to give other names for $\frac{3}{5}$ and $\frac{4}{7}$ as you list them on the board in strings of equalities.

$$\frac{3}{5} = \frac{6}{10} = \frac{9}{15} = \frac{12}{20} = \frac{15}{25} = \frac{18}{30} = \frac{21}{35} \ldots$$

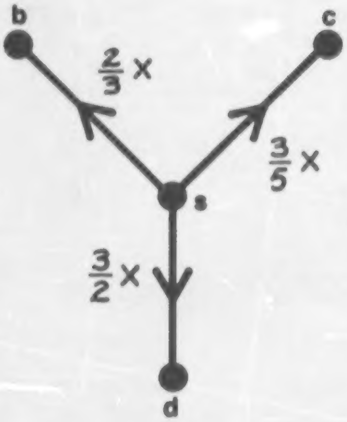
$$\frac{4}{7} = \frac{8}{14} = \frac{12}{21} = \frac{16}{28} = \frac{20}{35} \ldots$$

S: $\frac{3}{5} = \frac{12}{20}$ and $\frac{4}{7} = \frac{12}{21}$, so $\frac{3}{5}$ is larger than $\frac{4}{7}$.

S: $\frac{3}{5} = \frac{21}{35}$ and $\frac{4}{7} = \frac{20}{35}$, so $\frac{3}{5}$ is larger than $\frac{4}{7}$.

Exercise 4

Draw this arrow picture on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: The four dots are for whole numbers. Where is the largest number? Copy this picture and circle the dot for the largest number.

Let the students work independently for a few minutes.

S (pointing to d): Here is the largest number. Suppose a whole number is here (at s). $\frac{3}{2}$ of that number is larger, while $\frac{2}{3}$ of the number (at s) and $\frac{3}{5}$ of the number (at s) are both smaller.

Accept any explanation along these lines.

T: Draw a square around the smallest number in the picture.

S (pointing to c): This one.

T: Let's check by putting a number here (point to s). What do you suggest?

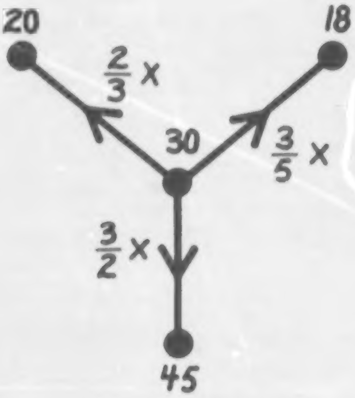
S: 30.

T: Why 30?

S: 30 is a multiple of 2, 3, and 5.

11 N

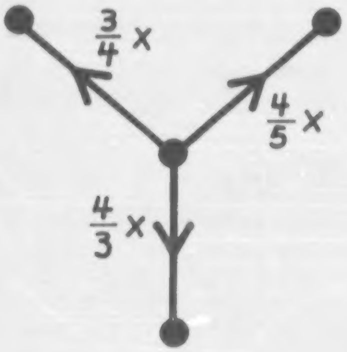
Label s "30" and ask the students to complete the arrow picture.



T: You were right. Here is the largest number (point to d) and here is the smallest number (point to c). Do you think the largest number and the smallest number will always be in these places (point to d and c)?

Accept any comment the students might make regarding the fact that no matter what positive number is at s, $\frac{3}{2}$ of the number is larger; and since $\frac{3}{5}$ is smaller than $\frac{2}{3}$, $\frac{3}{5}$ of the number at s is the smallest number.

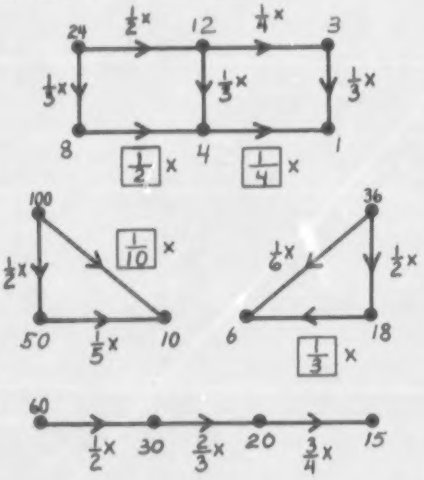
Repeat the activity above using this arrow picture.



Worksheets N62, N63, N64, and N65 are available.

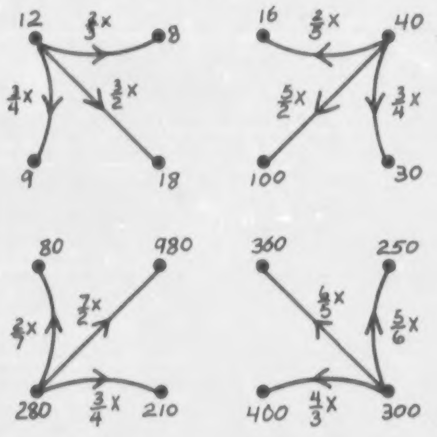
Name ____________ **N 62 ***

Label the dots and fill in the boxes for the arrows.


24
½ x
12
¼ x
3
⅓ x
⅓ x
⅓ x
8
[1/2] x
4
[1/4] x
1
100
[1/10] x
½ x
50
⅕ x
10
36
⅙ x
½ x
6
18
[1/3] x
60
½ x
30
⅔ x
20
¾ x
15

Name ____________ **N 63 ****

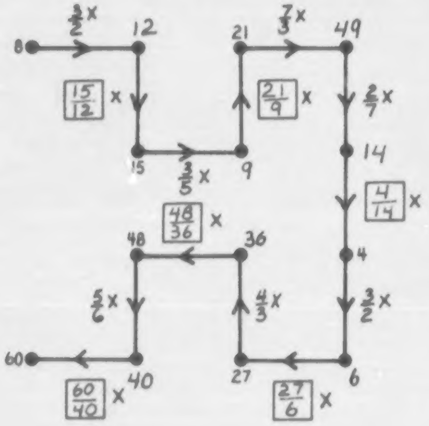
All the dots are for WHOLE numbers.
Label the dots. Many answers are possible.


12
⅔ x
8
¾ x
3/2 x
9
18
16
⅖ x
40
5/2 x
¾ x
100
30
80
980
2/7 x
7/2 x
280
¾ x
210
360
250
6/5 x
⅚ x
400
4/3 x
300

(Many other solutions are possible.)

Name ____________ **N 64 *****

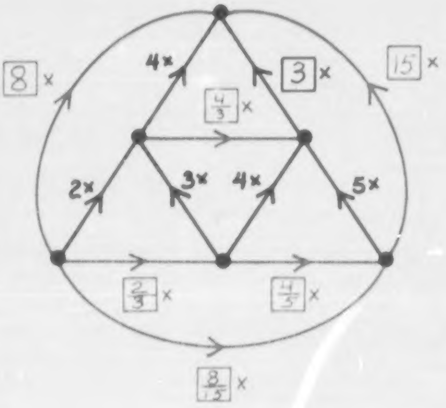
Label the dots and fill in the boxes for the arrows.


8
3/2 x
12
[15/12] x
15
⅗ x
9
21
[21/9] x
7/3 x
49
2/7 x
14
[4/14] x
4
3/2 x
6
[27/6] x
27
4/3 x
36
[48/36] x
48
⅚ x
40
60
[60/40] x

(The answers above are those that may be given by students. It is possible that other correct names could be given instead, e.g., $\frac{15}{12} = \frac{5}{4} = \frac{10}{8} = \frac{20}{16} = \ldots$)

Name ____________ **N 65 ******

Fill in the boxes for the arrows.

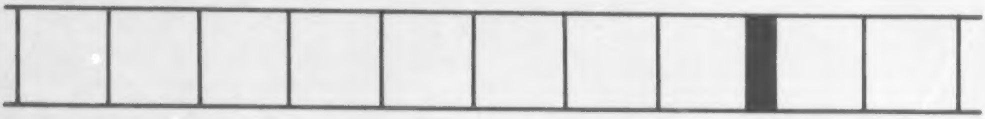

[8] x
4x
[3] x
[15] x
[4/3] x
2x
3x
4x
5x
[2/3] x
[4/3] x
[8/15] x

19 N

7.3 A New Name for One-Third

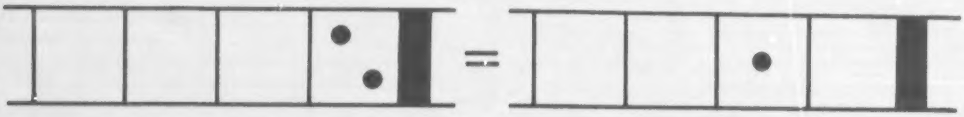
Exercise 1

Draw this part of an abacus on the chalkboard or use Minicomputer boards. The gray bar should be drawn in green on the chalkboard.



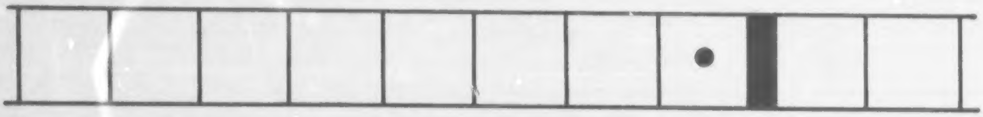
T: Do you remember the binary abacus and its rule?

Invite a student to explain the binary abacus rule to the class. In the unlikely event that no student recalls the binary abacus rule, you should remind the class that the rule is:



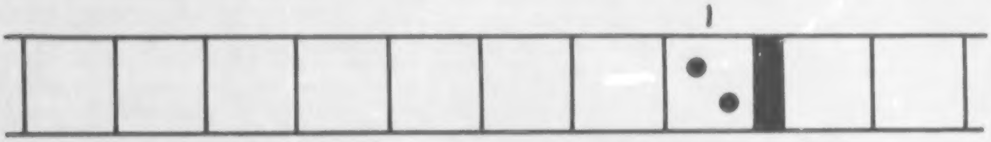
> Two checkers on a board represents the same number as one checker on the next board to the left.

Draw (put) one checker on the abacus as shown here.



T: This is the number 1.

Label the board "1" and add one checker to the ones' board.

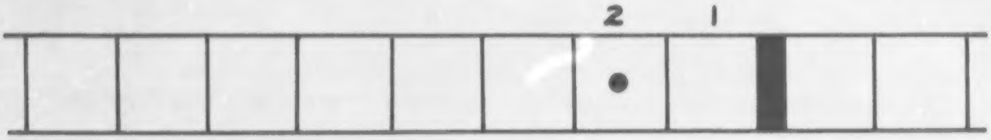


T: What number is this?

S: 2.

T: Can you put 2 on the abacus using exactly one checker?

Ask a student to put 2 on this abacus with exactly one checker and then label the board "2".

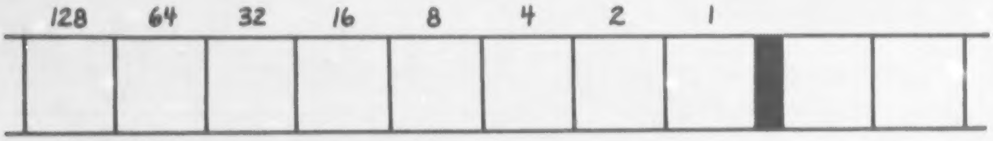


Point to the next board to the left and ask:

T: What board is this?

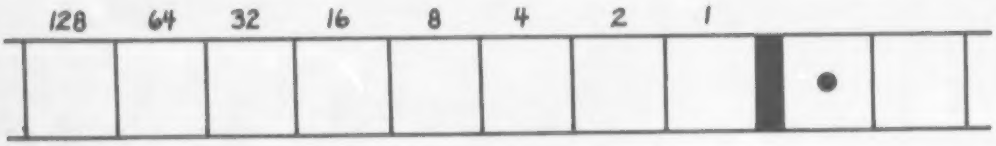
S: Fours' board.

Continue until you have labeled each board to the left of the (green) bar of the abacus that you have drawn.



19 N

Put one checker on the abacus as shown here.



T: What number is this?

If there are a few students who have difficulty with this situation, add another checker to the same board and ask:

T: What number is this?

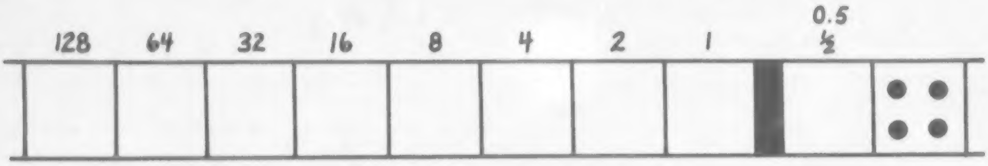
If necessary, ask a student to make a play to show that the number is 1.

S: 1.

T: Two checkers on this board is the number 1, so one checker on this board is ... ?

S: $\frac{1}{2}$ or 0.5.

Label the board both "0.5" and "$\frac{1}{2}$"; then display this configuration of checkers.



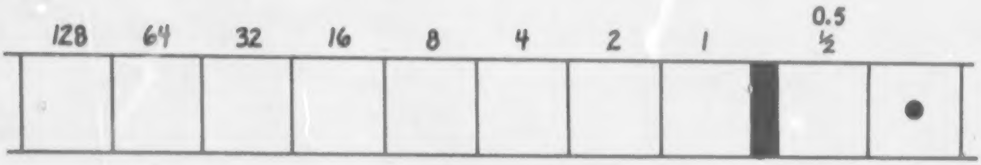
T: What number is this?

S: 1.

Ask a student to make the plays to show that the number is 1.

T: If I put one checker on this same board, what number is it?

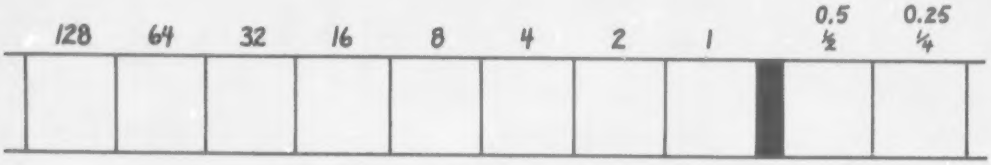
Put one checker on the abacus as shown below.



S: $\frac{1}{4}$.

S: 0.25.

Label the board both "$\frac{1}{4}$" and "0.25" and extend the abacus one board to the right.



T (point to the new board): How many checkers will it take on this board to have the number 1?

S: 8.

Put one checker on the new board.

T: If 8 checkers on this board is the number 1, then what is this number?

S: $\frac{1}{8}$.

19 N

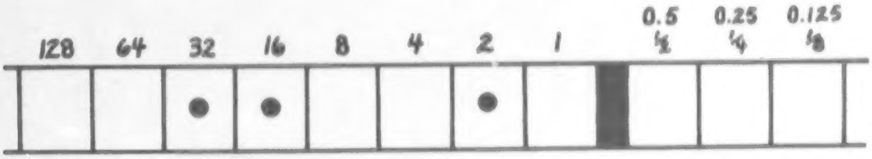
T: Who knows the decimal name for $\frac{1}{8}$?

You can give the class a hint by naming 0.5 and 0.25 as "0.500" and "0.250", respectively.

S: 0.125.

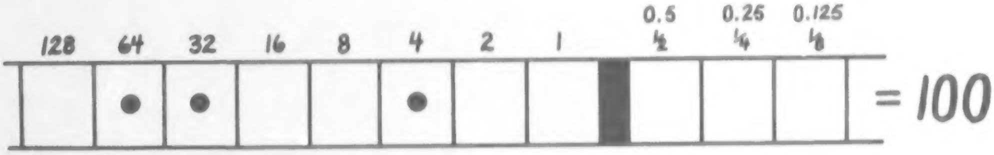
Remove the checkers from the abacus and ask the students to draw an abacus on their paper and to put 50 on using at most one checker on any board.

After a minute or two, ask a student who is finished to put 50 on the demonstration abacus using at most one checker on any board.


128
64
32
16
8
4
2
1
0.5
½
0.25
¼
0.125
⅛

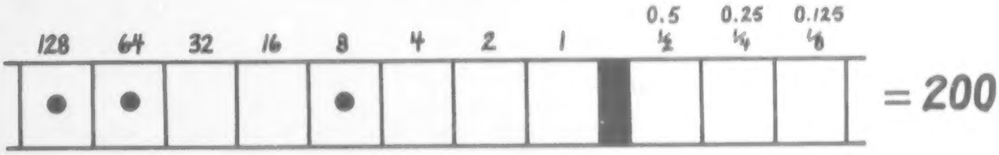
T: Now quickly, who can put 100 on the abacus?

S: Move each checker one board to the left.


128
64
32
16
8
4
2
1
0.5
½
0.25
¼
0.125
⅛
= 100

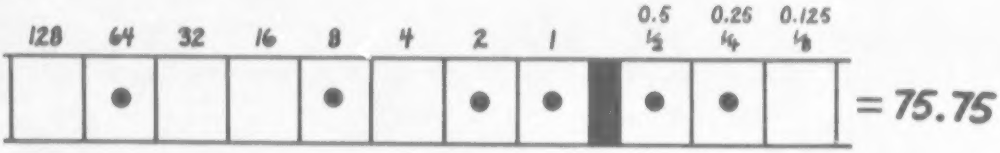
T: Now 200, very quickly.

S: Move each checker one board to the left.


128
64
32
16
8
4
2
1
0.5
½
0.25
¼
0.125
⅛
= 200

T: Draw another abacus on your paper and put 75.75 on it. Remember, use at most one checker on a board.

After a few minutes of working independently, ask a student who is finished to put 75.75 on the demonstration abacus using at most one checker on a board.



T: I will move each checker one board to the left. How can you very easily calculate the new number?

S: Multiply by two, because moving the checkers one board to the left doubles the number.

Write this calculation on the board.

$$\begin{array}{r} 75.75 \\ \times \quad 2 \\ \hline \end{array}$$

T: Let's do the calculation mentally. 2 x 75 is ... ?

S: 150.

T: 2 x 0.75?

S: 1.50.

T: Now, 2 x 75.75?

S: 151.50.

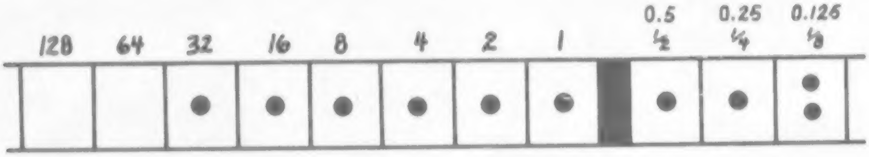
19 N

Write the product on the board.

$$\begin{array}{r} 75.75 \\ \times \quad 2 \\ \hline 151.50 \end{array}$$

T: I will put a new number on the abacus using many checkers, but I think you can calculate it very easily, if you think about what I am doing. Write this number on your paper.

Put this configuration on the abacus.



S: 64.

Let a student make the plays for verification.

Exercise 2

T: It is very easy to put $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, and so on on the binary abacus. Now I would like to put $\frac{1}{3}$ on the abacus, but I am not so sure it will be as easy. Why is it easy to put $\frac{1}{4}$ on the abacus?

S: Because there is a one-quarters' board. Four checkers on that board is the same as one checker on the ones' board.

T: How can we put $\frac{1}{3}$ on the abacus?

Let the students make suggestions. Try to pursue their suggestions as much as possible.

In the following dialogue, we describe how the solution of this problem developed in one of our development classes.

S: Put $\frac{1}{4}$ and $\frac{1}{8}$ on the abacus.

Write this equation on the board.

$$\frac{1}{4} + \frac{1}{8} = \square$$

T: Can you suggest another way to name $\frac{1}{4} + \frac{1}{8}$?

S: $\frac{2}{8} + \frac{1}{8} = \frac{3}{8}$.

Put this information on the board.

$$\frac{1}{4} + \frac{1}{8} = \frac{2}{8} + \frac{1}{8} = \frac{3}{8}$$

S: $\frac{3}{8}$ is close to $\frac{1}{3}$, because $\frac{1}{3} = \frac{3}{9}$.

T: Which is larger, $\frac{1}{3}$ or $\frac{3}{8}$?

S: $\frac{3}{8}$, because $\frac{3}{8}$ is larger than $\frac{3}{9}$.

NOTE: Another way to examine the inequality is to compare the decimal names for $\frac{1}{3}$ and $\frac{3}{8}$.

T: What is a decimal name for $\frac{1}{8}$?

S: 0.125.

T: What is a decimal name for $\frac{3}{8}$?

19 N

S: 0.375.

T: What is a decimal name for $\frac{1}{3}$?

If there are some students who have difficulty with this, ask:

T: Do you remember 1,000's Dream? What was another name for 1,000?

S: 999.999 ....

T: Suppose 1 had been playing 1,000's game. What would be another name for 1?

S: 0.999 ....

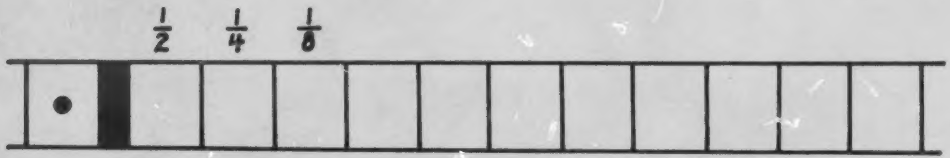
S: So $\frac{1}{3}$ = 0.333 ....

T: The number on the abacus is too big.

S: Move the checker on the one-eighths' board one board to the right.

T: That's a good idea, but I would like to suggest another idea about how one can put $\frac{1}{3}$ on the abacus. Let's start with 1.

Draw this binary abacus on the chalkboard and put a checker on the ones' board.

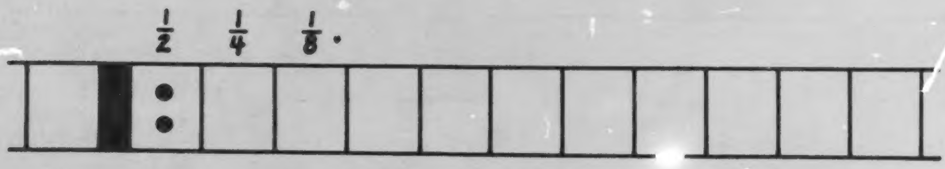


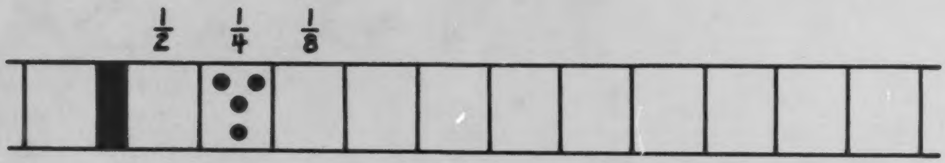
T: Now we will make some plays so that we can easily find $\frac{1}{3}$ of 1. What should we do?

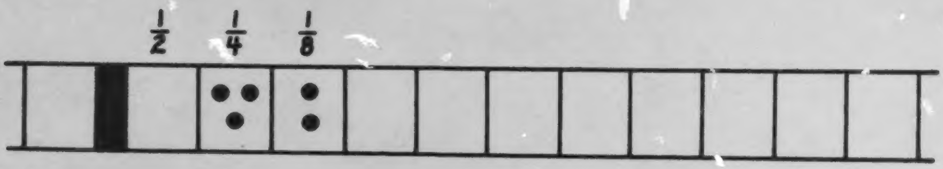
S: Get three checkers on each board that has checkers on it.

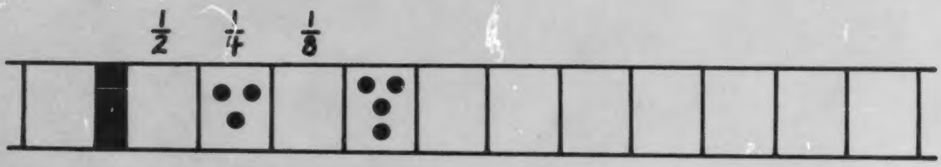
T: Very good. We want to calculate "$\frac{1}{3}$ of 1" on this binary abacus.

Begin making the sequence of plays shown below. When it is obvious that some students have discovered the pattern, have them make the plays.





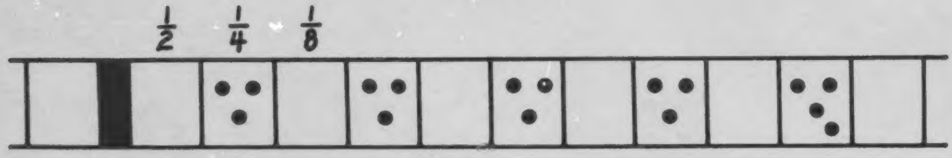




and so on.

19 N

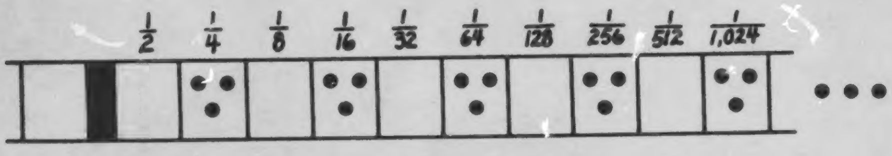
Continue until this configuration is on the abacus.



T: Will this ever end?

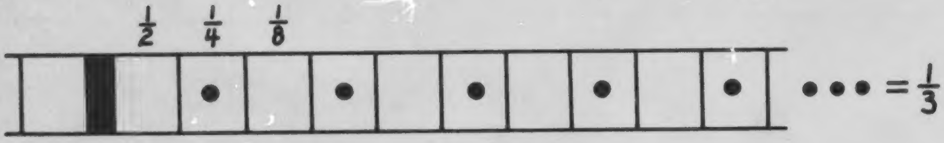
S: No, we can go on and on making backward plays.

Indicate the repeating pattern by removing the extra checker and writing three dots as illustrated below.


1/2
1/4
1/8
1/16
1/32
1/64
1/128
1/256
1/512
1/1,024
• • •

T: Now, who can show us $\frac{1}{3}$?

Let a student remove two out of every set of three checkers, leaving this configuration on the abacus.



Write this number sentence on the board.

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1{,}024} + \cdots$$

T: Remember, the addition goes on and on.

Exercise 3

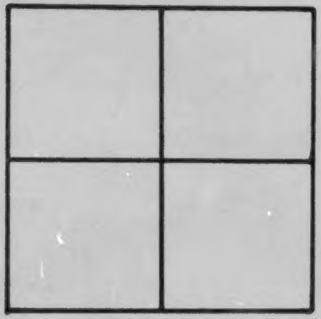
T: I want to tell you a story about four children – Bruce and three of his friends. You probably remember Bruce from some other stories. The four children wanted to divide a cake equally among themselves.

Draw this square "cake" on the board. The region will be subdivided several times as the story unfolds, so make the square as large as possible. Then it will be easier to view the subdivisions.

T: Show us how the cake could be divided so that each child gets an equal amount.

Invite a student to divide the cake. Encourage the student to divide it into four squares.

NOTE: There are many other configurations that are correct but you want this one because it will be easier to follow the remainder of the problem.



T: What fraction of the entire cake did each child receive?

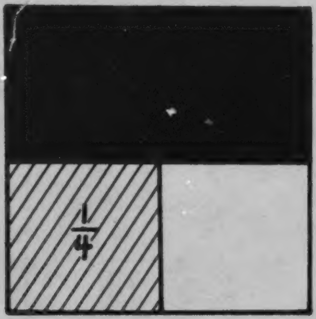
S: One-fourth.

19 N

Label three portions "$\frac{1}{4}$".

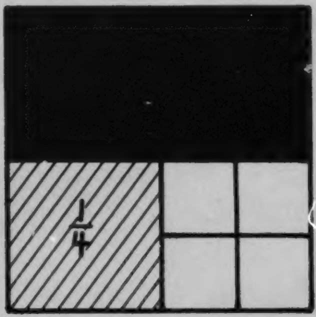
T: I will shade one portion red, one portion blue and one portion yellow (or any three different colors you choose). I will not shade Bruce's portion.

NOTE: In the drawings that follow, the yellow portions are hatched.



T: Bruce said, "I am not very hungry. I will share my portion equally among the four of us."

Let a student make the division of Bruce's portion, again into square subdivisions.

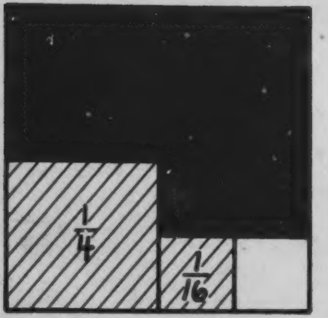


T: What additional fraction of the whole cake did each child receive?

S: One-sixteenth (one-fourth of one-fourth).

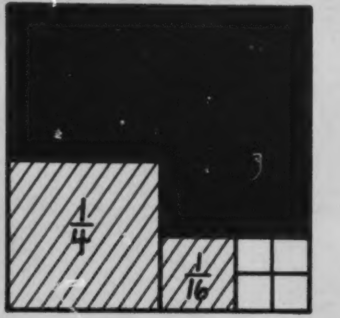
Label three portions "$\frac{1}{16}$".

T: I will shade three of the portions as before.



T: Again Bruce said, "My piece of cake is still too big! I will share my portion equally among the four of us."

Let a student make the division.

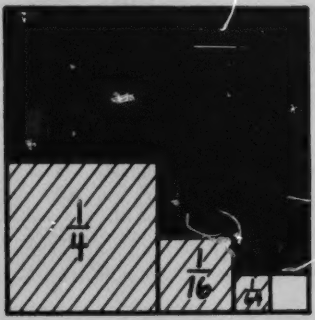


T: What additional fraction of the whole cake did each child receive?

S: $\frac{1}{64}$.

Label three portions "$\frac{1}{64}$".

T: I will shade three of the portions as before.



19 N

Continue the story with Bruce protesting that his portion is still too large. Each time let a student make the division. Be sure to name the fraction and shade the appropriate regions. Repeat this several times more until it is too difficult to make the divisions.

T: We will stop the drawing, but our story goes on and on.

What fraction of the whole cake would be in red?

S: $\frac{1}{4} + \frac{1}{16} + \frac{1}{64}$ and so on.

S: $\frac{1}{3}$.

If the latter answer is not suggested by the class, ask about how much of the cake is shaded red.

Write this equality on the board.

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1{,}024} + \cdots$$

T: Do you see any connection between the story and the problem of the abacus?

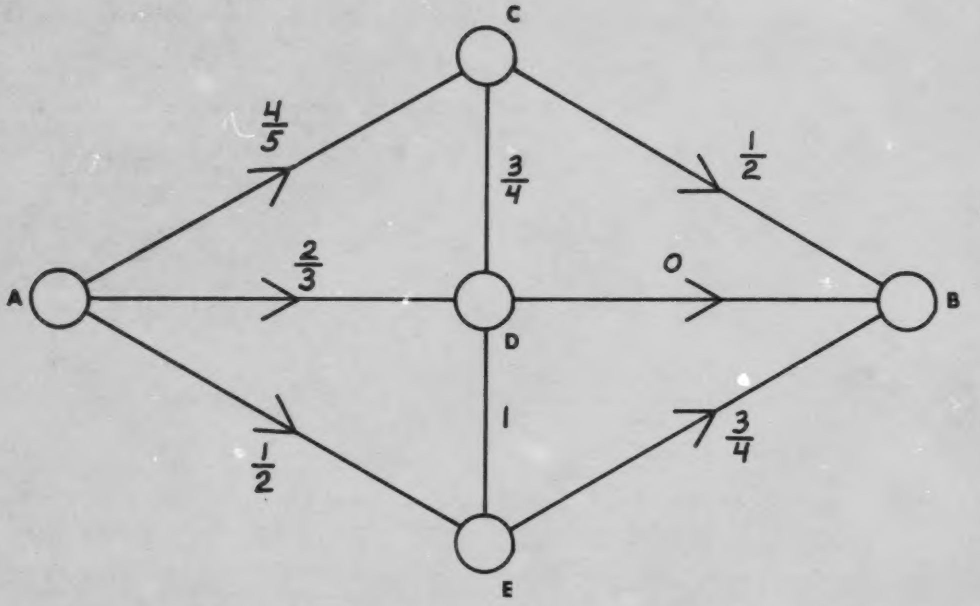
S: They are the same.

S: It's two different ways to look at the same problem.

Accept any reasonable comments the students might make.

7.4 Dangerous Roads

Draw this picture on the board.



T: This is a map of a portion of the wild-wild West more than 100 years ago. The map shows a road system between two towns — A and B. The arrows indicate one-way roads and the cords indicate two-way roads. Can you trace a path that a traveler could take from A to B?

Let several students trace paths from A to B using different colors of chalk for different paths. A student might suggest a path which includes doubling back and forth between C and D (or D and E). If such a suggestion is made, remind the class that due to the dangers involved in travel, a more direct route is desirable. One path should not include the same road more than once.

T: We all have heard that in the wild West travel was dangerous; the roads were infested with dangerous animals, bandits, and murderers. What do you think the numbers in the picture represent?

Let the class express their ideas. Someone will very likely make a suggestion along these lines.

S: The chance that a traveler will survive going from one town to another.

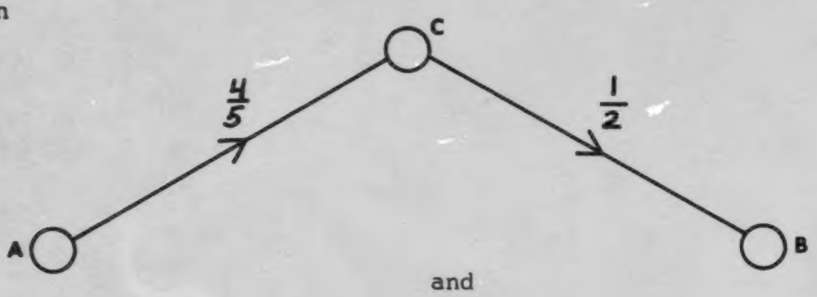
T: Yes. For example, if you travel on this road labeled "$\frac{4}{5}$" from A to C, it has been observed that you have about 4 chances out of 5 of surviving. Suppose 100 people traveled the road labeled "$\frac{4}{5}$". About how many of these people would you expect to arrive at the next town?

S: About 80, because $\frac{4}{5} \times 100 = 80$.

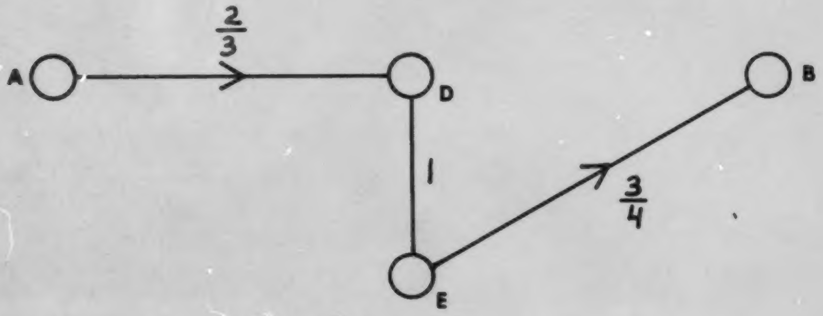
If necessary, remind the students that to calculate "$\frac{4}{5} \times 100$", they can divide 100 by 5 and then multiply the result by 4.

T: Here is your problem. Suppose you must make a trip from A to B. Which route would you choose? Which is the safest route?

Listen to the students' explanations. If there are different choices, write all of them on the board; e.g., in our development classes, students were split between



and



It is likely someone will observe that the road labeled "0" from D to B should never be traveled as the chance of survival is 0! If this observation is not made by any students, ask:

T: Are there any roads which you would avoid?

S: The road from D to B. No one survives on that road.

T: At least, no one has been known to survive. How many paths contain the road from D to B?

S: Three paths.

Ask students to trace the three paths.

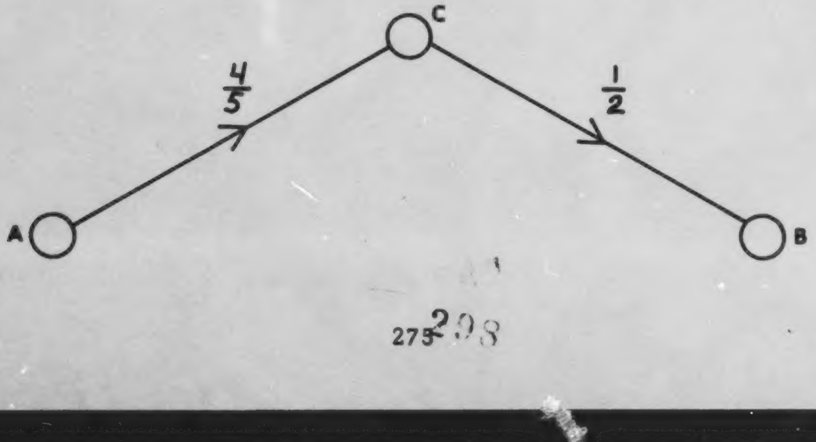
A to C to D to B

A to D to B

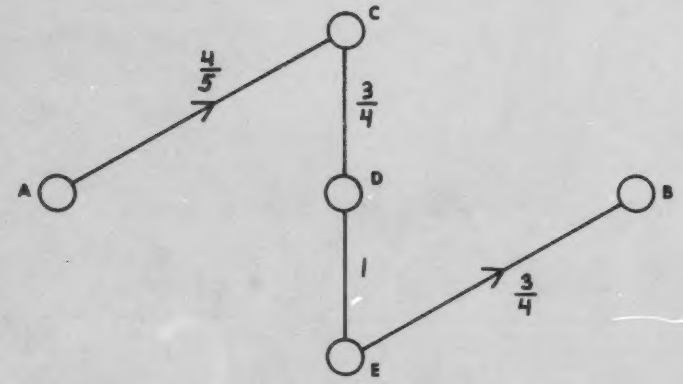
A to E to D to B

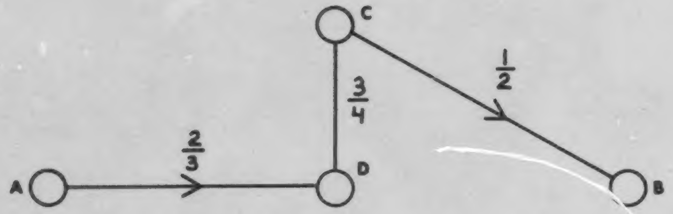
T: How many more paths are there from A to B, and which path is the safest to travel? Remember, we want to avoid the road from D to B.

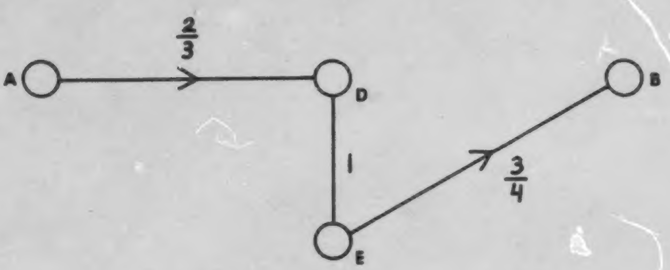
The class will probably suggest that the paths be traced. In that case, record each path (there are six more paths) as it is traced on the board. This should be done aside from the picture so they are easily distinguished and can be discussed individually. The six paths are shown below and on the next page.

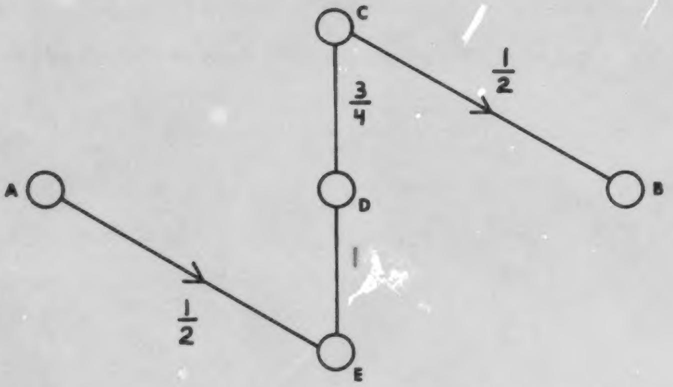


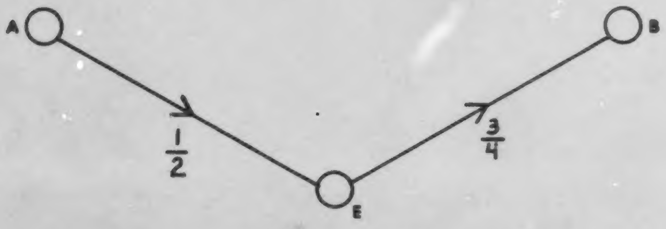
21N


C
4/5
3/4
A
D
B
1
3/4
E


C
1/2
3/4
2/3
A
D
B


2/3
A
D
B
1
3/4
E


C
1/2
3/4
A
D
B
1
1/2
E

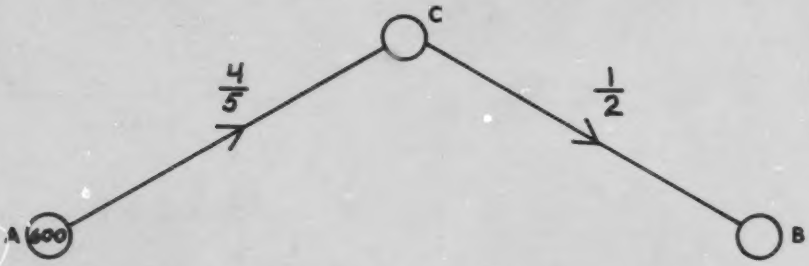

A
B
1/2
3/4
E

T: Six paths altogether. How can we determine the safest path to take?

S: We could imagine that many people traveled along the paths and see for which path we would expect the most survivors.

T: How many people do you suggest we use?

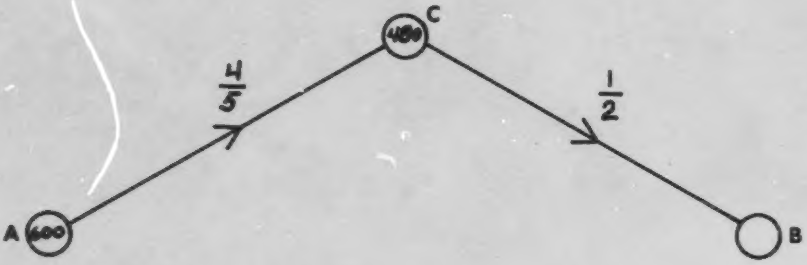
Accept any reasonable number you wish; however, multiples of 120 will keep the calculations simple. For the purpose of describing this part of the lesson, suppose 600 is chosen. Start with a path suggested earlier as being safest.



T: Let's examine this path from town A to C and then to B. Suppose 600 people start at A. About how many will arrive at C? Why?

S: About 480, because $\frac{4}{5} \times 600 = 480$.

Add this information to your drawing.

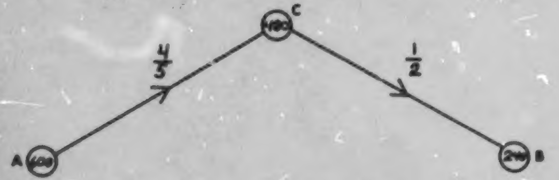


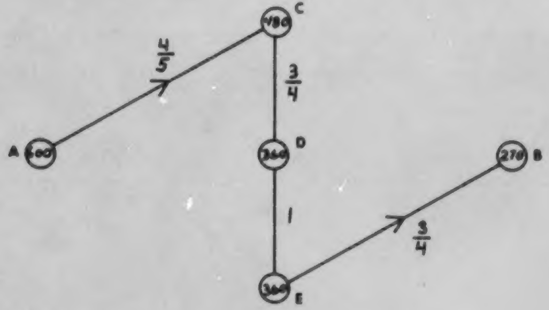
T: About how many people will arrive safely at B?

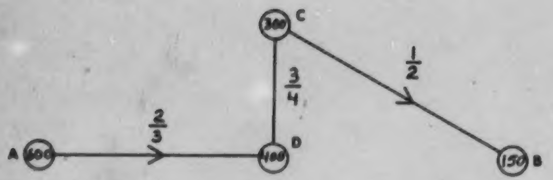
S: About 240, because $\frac{1}{2} \times 480 = 240$.

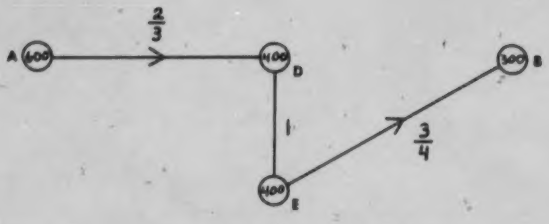
21 N

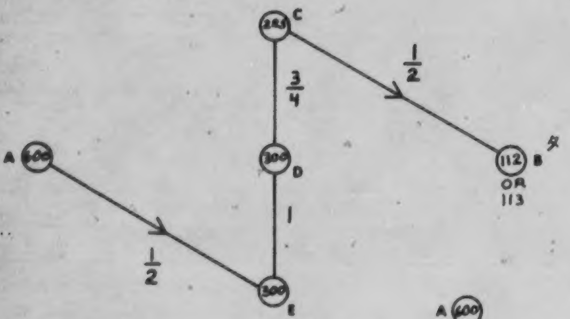
Continue in this manner, analyzing the five remaining paths. Correct answers for all six paths are shown below.

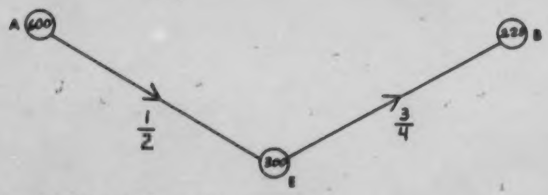

C 480
$\frac{4}{5}$
$\frac{1}{2}$
A 600
240 B


C 480
$\frac{4}{5}$
$\frac{3}{4}$
A 600
360 D
270 B
1
$\frac{3}{4}$
360 E


300 C
$\frac{1}{2}$
$\frac{3}{4}$
$\frac{2}{3}$
A 600
400 D
150 B


$\frac{2}{3}$
A 600
400 D
300 B
1
$\frac{3}{4}$
400 E


225 C
$\frac{1}{2}$
$\frac{3}{4}$
A 600
300 D
112 B *
OR 113
1
$\frac{1}{2}$
300 E


A 600
225 B
$\frac{1}{2}$
$\frac{3}{4}$
300 E

---

* $\frac{1}{2} \times 225 = 112.5$, so about 112 or 113 people will survive.

S: So, the path A to D to E to B is the safest — about 300 people out of 600 survive.

T: If you were to take this route, what are the chances of your survival?

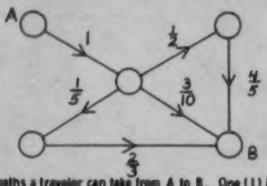
S: 1 out of 2 or $\frac{1}{2}$.

T: So 1 chance out of 2 is the best we can do.

Worksheets N66 and N67 are available.

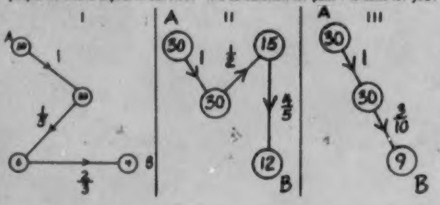
Name ____________________ **N66 ***

This is the map of the roads from town A to town B.


A
1
$\frac{1}{2}$
$\frac{1}{5}$
$\frac{3}{10}$
$\frac{4}{5}$
$\frac{2}{3}$
B

There are three paths a traveler can take from A to B. One (I) is shown below; draw the other paths (II and III).

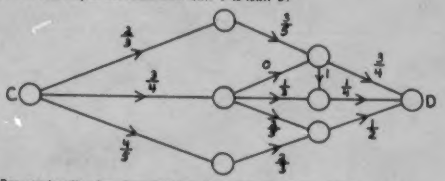
Suppose 30 people start from town A. For each path, calculate the number of people we would expect to survive. The calculation for path I is done for you.


I
A
1
$\frac{1}{5}$
$\frac{2}{3}$
B
II
A
30
1
$\frac{1}{2}$
30
15
$\frac{4}{5}$
12
B
III
A
30
1
30
$\frac{3}{10}$
9
B

Which path is the safest? II

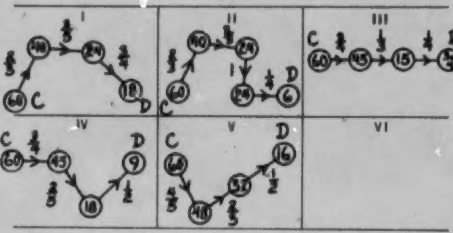
Name ____________________ **N 67 ****

This is the map of the roads from town C to town D.


C
$\frac{2}{3}$
$\frac{3}{5}$
$\frac{3}{4}$
0
1
$\frac{1}{3}$
$\frac{1}{4}$
$\frac{3}{4}$
$\frac{1}{3}$
$\frac{4}{5}$
$\frac{2}{3}$
$\frac{1}{2}$
D

Draw each path a traveler might take from C to D. If there is a road that should definitely be avoided, do not include it in any of the paths you draw.

Suppose 60 people start from C. For each path, calculate the number of people we would expect to survive.

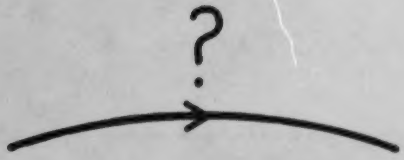

I
C
60
$\frac{2}{3}$
40
$\frac{3}{5}$
24
$\frac{3}{4}$
18
D
II
C
60
$\frac{2}{3}$
40
$\frac{3}{5}$
24
1
24
$\frac{1}{4}$
6
D
III
C
60
$\frac{3}{4}$
45
$\frac{1}{3}$
15
$\frac{1}{4}$
D
IV
C
60
$\frac{3}{4}$
45
$\frac{2}{5}$
18
$\frac{1}{2}$
9
D
V
C
60
$\frac{4}{5}$
48
$\frac{2}{3}$
32
$\frac{1}{2}$
16
D
VI

Which path is the safest? I

7.5 Guess My Rule

Exercise 1

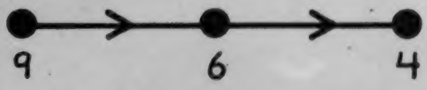
Draw this red arrow on the board.



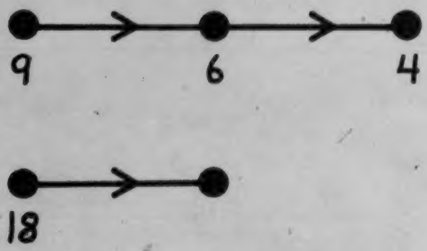
T: This red arrow is for my secret rule. I will give you some hints and you should try to guess what the red arrow is for.

NOTE: The red arrow is for $\frac{2}{3}x$, but do not tell the students.

Slowly draw these two arrows on the board.



Add this arrow to your picture.



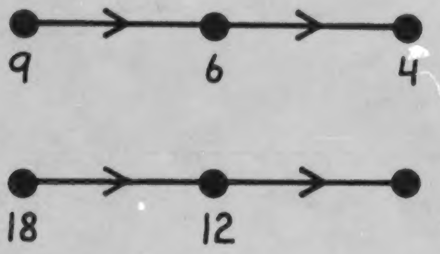
T (pointing to the unlabeled dot): Do you know my rule yet? Which number will I put here?

24 N

Let the students express their opinions. If someone suggests 12, then label the dot "12". If no one suggests 12, label the dot "12" as you say:

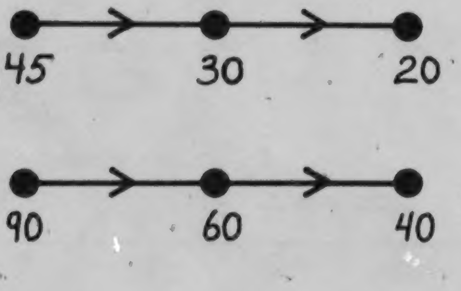
T: You don't know my rule yet. I will tell you 12 is here. Think about what the red arrows could be for.

Add another arrow to your picture.



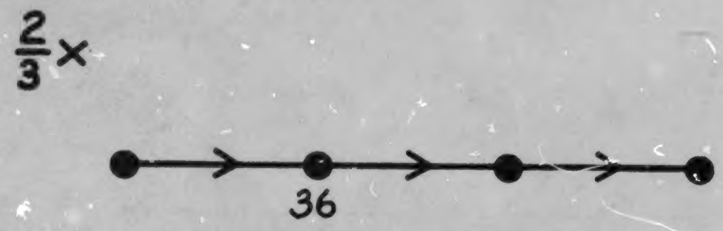
T: On your paper, write your guess for which number I will put here (point to the unlabeled dot).

Continue in this manner, slowly and progressively drawing these arrow roads on the board. For each arrow road, label the left-most dot and invite students to label the other dots.



By this time, many students will have guessed the rule. Ask someone to explain that the red arrows are for $\frac{2}{3}$x.

Draw this arrow picture on the board.

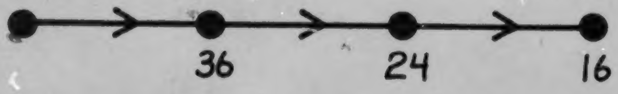


T: How do you calculate "$\frac{2}{3} \times 36$"?

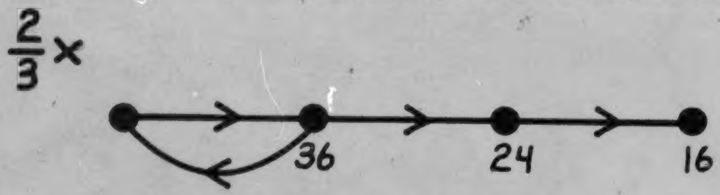
S: $36 \div 3 = 12$ and $2 \times 12 = 24$, so $\frac{2}{3} \times 36 = 24$.

T: Copy the arrow picture and label the dots.

Let the students work independently for a few minutes, and then check the work collectively. A partially labeled arrow road is shown below.

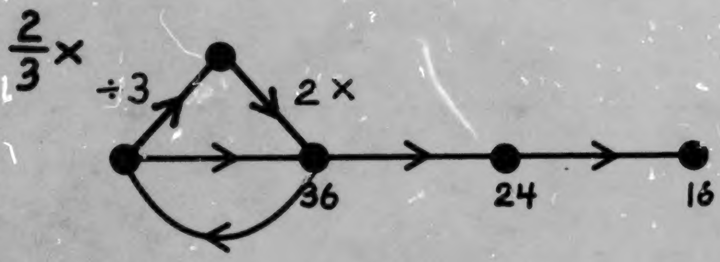


Some students might find that labeling this last dot is difficult. If several students experience difficulty, add a return arrow to the picture.



T: What could this return arrow be for?

If no one suggest $\frac{3}{2}$ ×, then continue by drawing these two arrows.



T: $\frac{2}{3}$ × is the same as ÷ 3 (trace the ÷3 arrow) followed by 2× (trace the 2× arrow).

Trace against the arrows saying:

T: Against 2× followed by against ÷3. What could the return arrow be for?

S: ÷2 followed by 3×, so the return arrow is for $\frac{3}{2}$ ×.

T: Calculate "$\frac{3}{2}$ × 36"? Write it on your paper.

Allow a minute or two for the students to do the calculation.

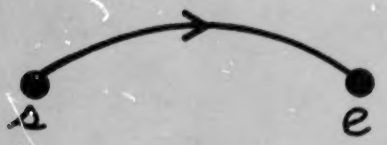
S: $36 \div 2 = 18$ and $3 \times 18 = 54$, so $\frac{3}{2} \times 36 = 54$.

Label the dot "54".

Exercise 2

This exercise develops in a manner similar to Exercise 1. The rule to be guessed, however, is more challenging.

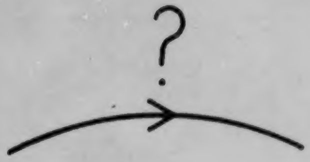
Let "s" be a starting number for the red arrow and "e" be the corresponding ending number.



For your information, the rule for the red arrow is given below. Do not write this rule on the board at this time.

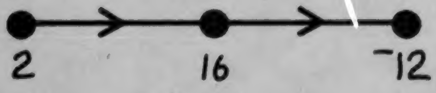
$$\boxed{(2 \times s) + e = 20}$$

Draw this red arrow on the board.

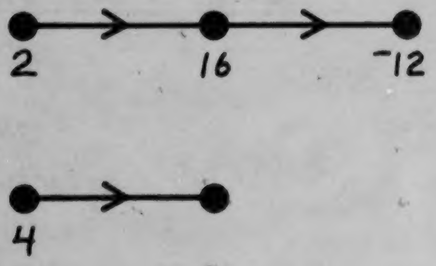


T: I have a new rule for you to guess.

Slowly draw these two arrows on the board.



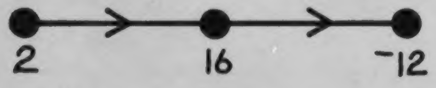
Add this arrow to your picture.

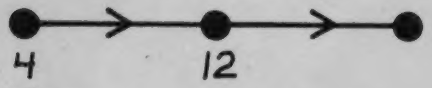


Let the students have an opportunity to guess what the ending number could be. If no one guesses it, label the dot "12". If someone guesses correctly, say:

T: Very good. Maybe you know what my rule is. Don't tell if you do, see if you can use it to label another dot.

Add another arrow to your picture.

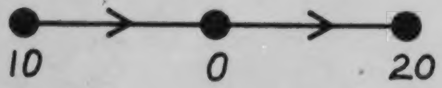




T: Write your guess for this number on your paper.

Ask the students to tell you their guesses for the ending number. If no one suggests ⁻4, then label the dot "⁻4". If some students do guess ⁻4, again ask them not to give away your secret. Continue drawing the road shown below. At this point, you may wish to give a hint to the class that your rule has something to do with 20.

Draw this arrow road slowly and encourage the students to think about the hint.



Ask a student to explain your rule.

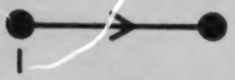
If no one has guessed the rule, tell it to the students yourself.

T: The rule for the red arrow is:

> Two times the starting number plus the ending number equals twenty.

Write this equation on the board and below it a red arrow starting at 1.

$$(2 \times \Delta) + e = 20$$



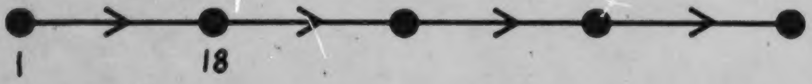
T (pointing to the dot labeled 1): Double the starting number.

S: 2 × 1 = 2.

T: What must the ending number be? Why?

S: 18, because 2+ the ending number is 20 and 2 + 18 = 20.

Label the dot "18" and extend the road.

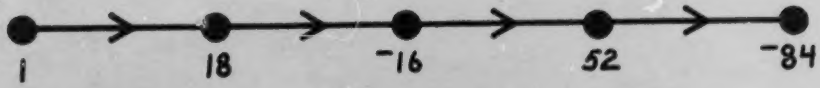


T: Copy this picture and label the dots.

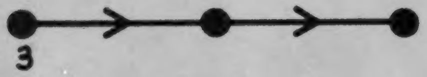
Let the students have several minutes to work while you walk around and monitor individual student progress. If you prefer, these problems could be solved collectively in a manner similar to the above.

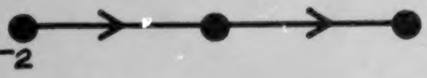
24 N

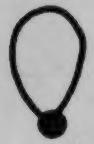
A completed picture is shown below.

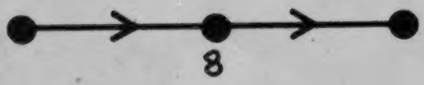


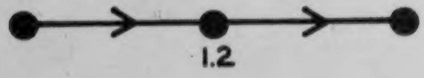
Draw these pictures on the board and ask the students to label the dots.





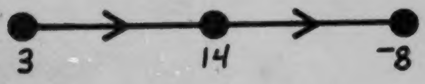


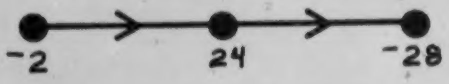


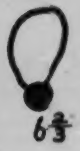


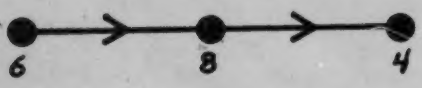
After the students have had several minutes to work on these problems, check them collectively.

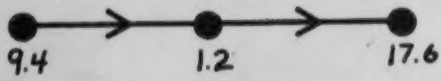
Completely labeled pictures are shown below.





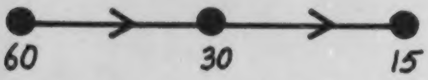




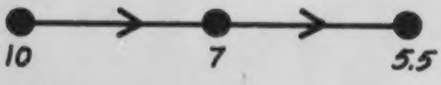


Exercise 3

Invite the students to invent their own rules for the class to guess. Use arrow pictures to record the results of applying rules. The following examples of rules are some that the students in our development classes invented.



Rule: $\frac{1}{2}x$



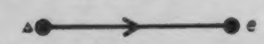
Rule: ÷ 2 followed by + 2

Worksheets N68, N69, and N70 are available.

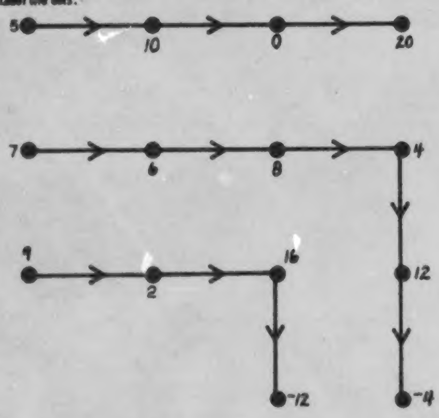
Name ____________________

N 68 *

$(2 \times \triangle) + e = 20$



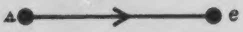
Label the dots.


5
10
0
20
7
6
8
4
12
-4
9
2
16
-12

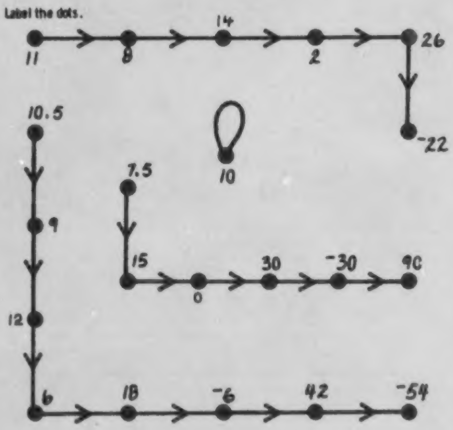
Name ____________________

N 69 **

$(2 \times \triangle) + e = 30$



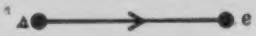
Label the dots.


11
8
14
2
26
-22
10
10.5
9
12
6
7.5
15
0
30
-30
90
18
-6
42
-54

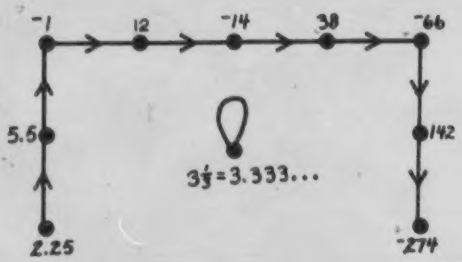
Name ____________________

N 70 ***

$(2 \times \triangle) + e = 10$



Label the dots.


-1
12
-14
38
-66
5.5
3⅓ = 3.333...
142
2.25
-274

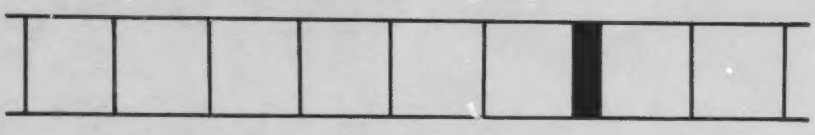
Flap is in this arrow picture. Flap is between 3 and 4.

Flap is $3\frac{1}{3}$.

7.6 A New Name for One - Half

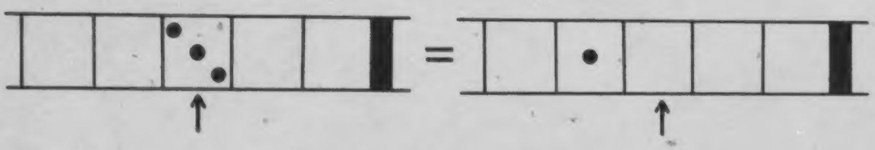
Exercise 1

Draw this part of an abacus on the chalkboard or use Minicomputer boards.



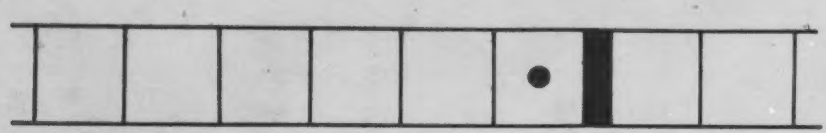
T: Do you remember the Base Three abacus and its rule?

Invite a student to explain the Base Three rule to the class. In the unlikely event that no student recalls the Base Three abacus, you should remind the class that the rule is:



> Three checkers on a board represents the same number as one checker on the next board to the left.

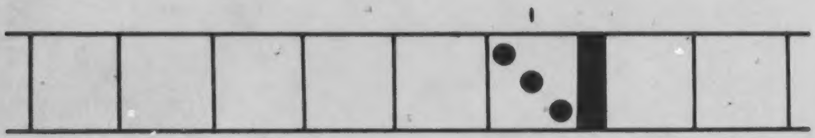
Draw (put) one checker on the abacus as shown here.



28 N

T: This is the number 1.

Label the board "1" and add two checkers to the ones' board.

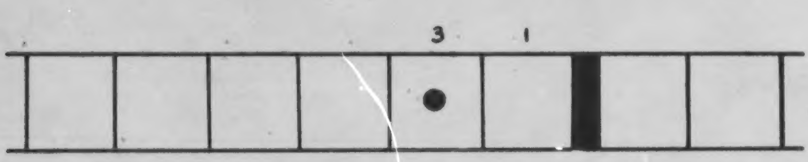


T: What number is this?

S: 3.

T: Can you put 3 on this abacus using exactly one checker?

Ask a student to put 3 on this abacus with exactly one checker and then label the board "3".

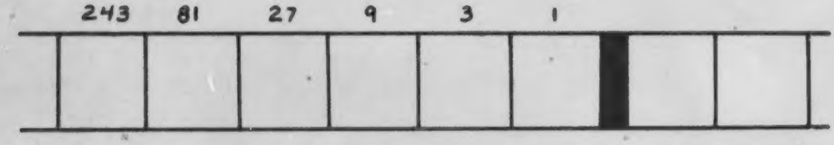


Point to the next board to the left and ask:

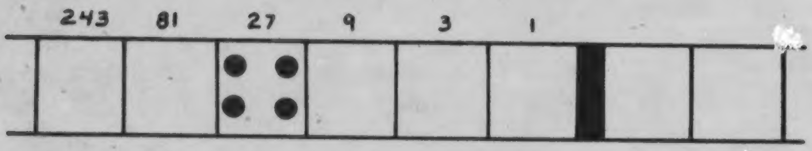
T: What board is this?

S: Nines' board.

Continue until you have labeled each board left of the (green) bar of the abacus that you have drawn.

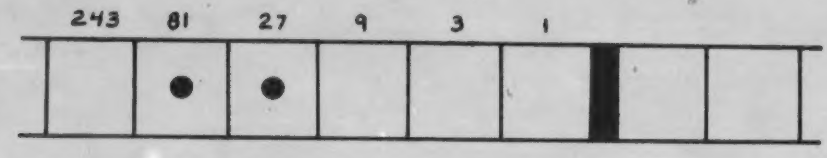


Put this configuration on the abacus.


243
81
27
9
3
1

T:	Show the same number using less than four checkers.

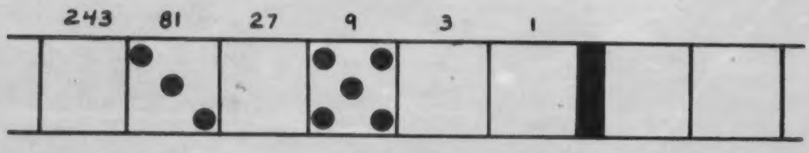
Call on a student to make a play to obtain:


243
81
27
9
3
1

T:	What number is this?

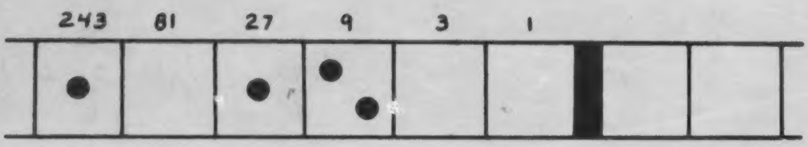
S:	108, because 81 + 27 = 108.

Display this configuration.


243
81
27
9
3
1

T:	Show the same number using fewer checkers.

Call on another student to make the plays to obtain:

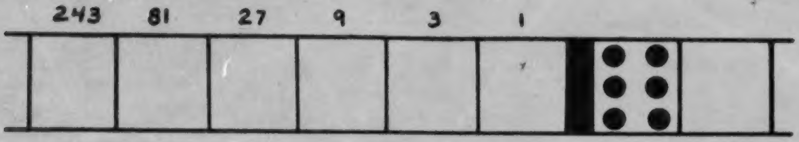

243
81
27
9
3
1

28 N

T: What number is this?

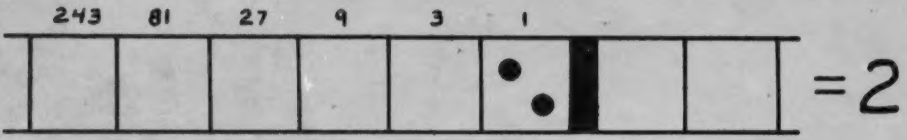
S: 288.

Next, put this configuration on the abacus.

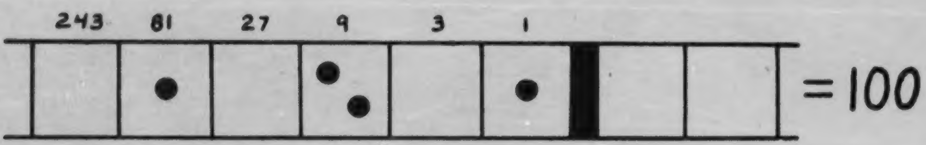


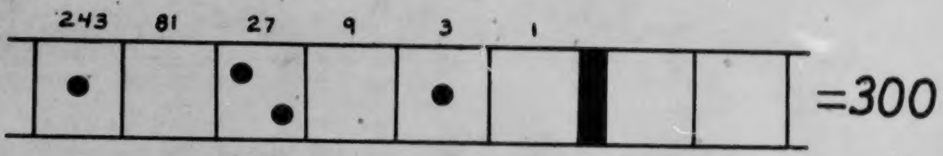
T: What number is this?

If necessary, call on a student to make the plays to obtain:

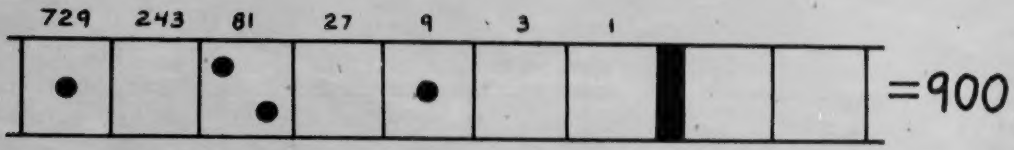


Distribute Worksheet N71. Let the students work independently on these problems for a few minutes. Check some of the problems collectively by asking students to show their solutions on the demonstration abacus. Encourage students to explain "short-cuts" for going from 100 to 300, and from 300 to 900, and from 900 to 2,700. One "short-cut" that might be discovered makes use of the fact that moving each checker one board to the left triples the number. If one starts with a configuration for 100 and moves each checker of the configuration one board to the left, the new number is 3 x 100 = 300.



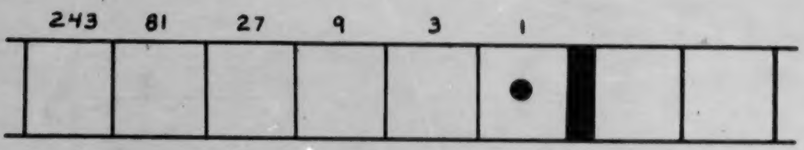

243
81
27
9
3
1
=300

Similarly, again move each checker one board to the left to obtain 900, because 3 x 300 = 900.


729
243
81
27
9
3
1
=900

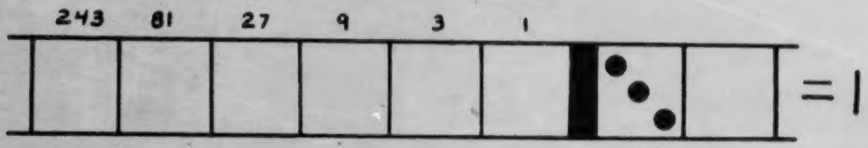
Exercise 2

Put 1 on the Base Three abacus.


243
81
27
9
3
1

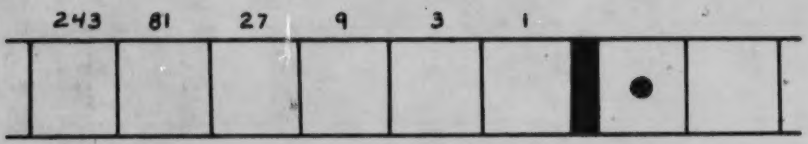
T: Can you put 1 on the abacus using three checkers?

Invite a student to make the backward play to obtain:


243
81
27
9
3
1
= 1

28 N

Remove two of the checkers.



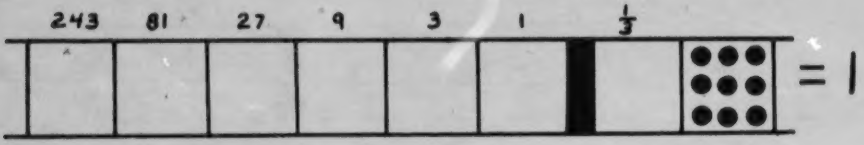
T: What number is this?

S: $\frac{1}{3}$, because $3 \times \frac{1}{3} = 1$.

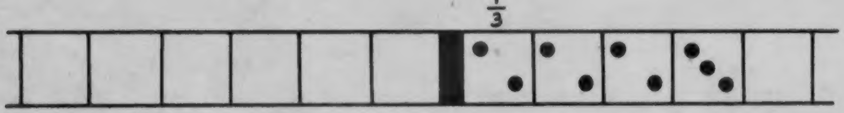
Label the board "$\frac{1}{3}$".

T: Can you put 1 on the abacus using nine checkers?

Give the class a minute to think about the problem, and then call on a student to show this solution.

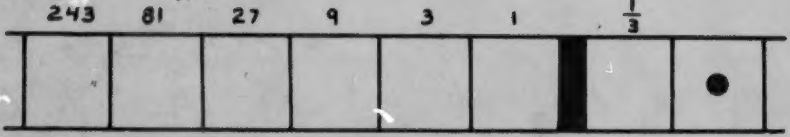


NOTE: Another solution which might be discovered is:



You should accept this solution, but ask that the first solution be put on the board.

Remove eight of the checkers ...

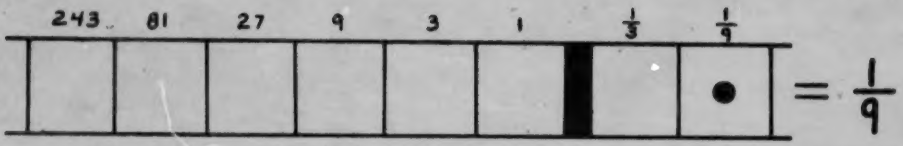

243
81
27
9
3
1
$\frac{1}{3}$

... and ask:

T: What number is this?

S: $\frac{1}{9}$, because $9 \times \frac{1}{9} = 1$.

Label the board "$\frac{1}{9}$".


243
81
27
9
3
1
$\frac{1}{3}$
$\frac{1}{9}$
$= \frac{1}{9}$

T: If there was another board to the right, what would it be for?

S: $\frac{1}{27}$.

T: And the next board?

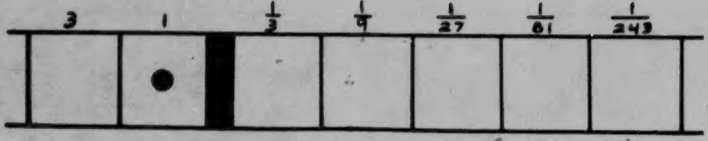
S: $\frac{1}{81}$.

T: And the next board?

S: $\frac{1}{243}$.

28 N

Again, put 1 on the Base Three abacus.

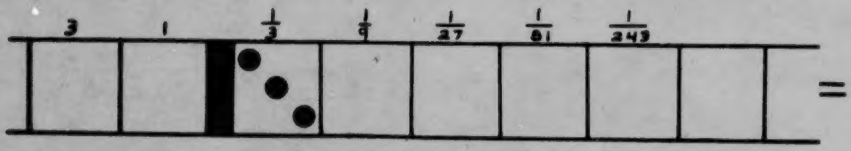

3
1
1/3
1/9
1/27
1/81
1/243

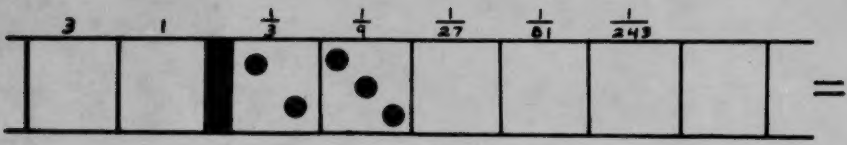
T: How can we put $\frac{1}{2}$ on this abacus?

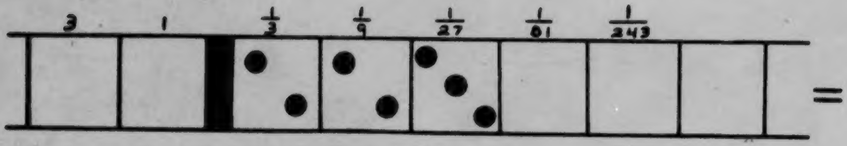
S: If we have groups of two checkers on the boards, then we could take one-half of the checkers off.

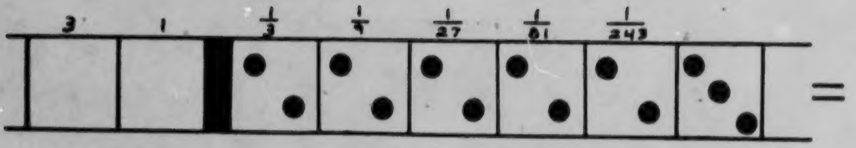
T: Can we get groups of two checkers?

Let the students have a few minutes to consider this problem. Encourage the students to make plays on the abacus. The students should suggest making backward plays like this:


3
1
1/3
1/9
1/27
1/81
1/243
=


3
1
1/3
1/9
1/27
1/81
1/243
=

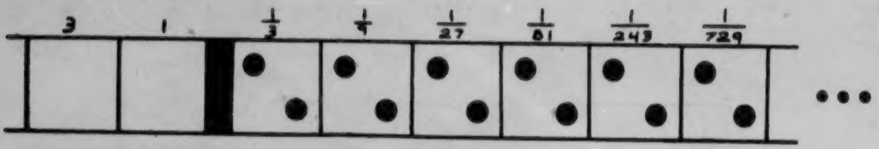

3
1
1/3
1/9
1/27
1/81
1/243
=


3
1
1/3
1/9
1/27
1/81
1/243
=

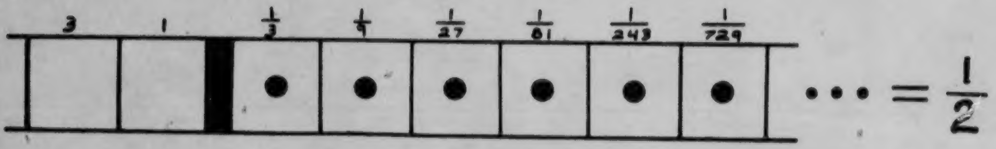
and so on.

T: Will this ever end?

S: No, it goes on and on just like in "Number 1,000's Dream".


3
1
1/3
1/9
1/27
1/81
1/243
1/729
...

T: Show us $\frac{1}{2}$.


3
1
1/3
1/9
1/27
1/81
1/243
1/729
... = 1/2

Write this number sentence on the board.

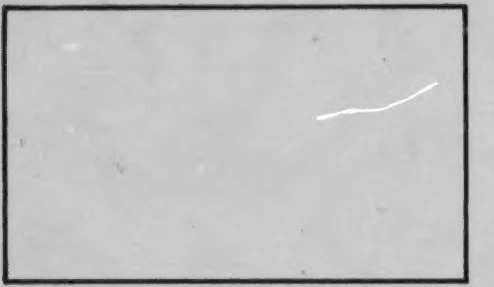
$$\frac{1}{2} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \cdots$$

Exercise 3

T: I want to tell you a story about three children: Mary, Allen, and Bruce. You probably remember Bruce from some other stories. The three children wanted to divide a cake equally among themselves.

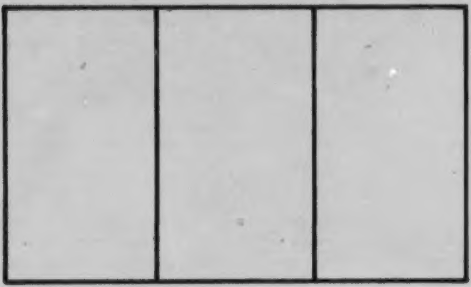
28 N

Draw this rectangular "cake" on the board. The region will be subdivided several times as the story unfolds, so make the rectangle as large as possible. Then it will be easier to make the subdivisions.



T: Show us how the cake could be divided.

Let a student show how to divide the cake. There are many correct subdivisions, but you should encourage this vertical division.

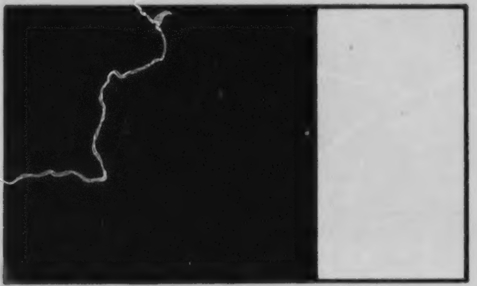


T: What fraction of the cake did each child receive?

S: One-third.

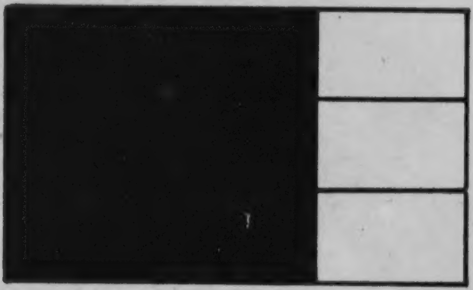
Label two portions "$\frac{1}{3}$".

T: I will shade Allen's portion blue and Mary's portion red.



T: Bruce said, "I am not very hungry. I will share my portion equally among the three of us."

Let a student make the division. This time encourage a horizontal division.

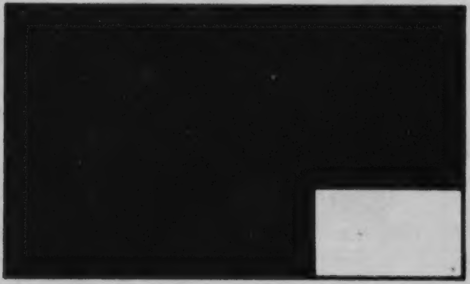


T: What additional fraction of the whole cake does each child receive?

S: One-ninth.

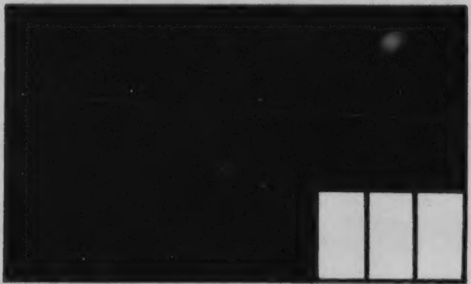
Label two portions "$\frac{1}{9}$".

T: I will again shade Allen's portion and Mary's portion.



T: Bruce said, "My piece of cake is still too big! I will share my portion equally among the three of us."

Let a student make the division. Encourage a vertical division.



T: What additional fraction of the whole cake does each child receive?

S: $\frac{1}{27}$, because altogether there are 27 small pieces (or one-third of one-ninth is one-twenty seventh).

Label two portions "$\frac{1}{27}$".

T: I will again shade Allen's portion and Mary's portion.

Continue the story with Bruce protesting that this portion is still too large. Each time, ask a student to make the division. Be sure to name the fraction and shade both Mary's and Allen's portion. Repeat this several times until it is too difficult to make any more divisions.

T: We will stop the drawing, but our story goes on and on. What fraction of the whole cake would be in red?

S: $\frac{1}{2}$.

S: $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243}$ and so on.

If the first answer is not suggested, ask about what portion of the cake is shaded red.

Write this equality on the board.

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \cdots = \frac{1}{2}$$

T: Do you see any connection between this story and last problem we did on the abacus?

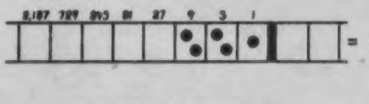
Accept any reasonable comments the students might make.

S: The story and the abacus problem are two different ways of looking at the same thing.

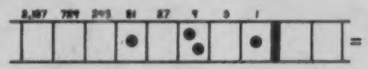
S: The fractions above the boards of the abacus are the same as those for the pieces of cake.

Name ____________ **N 71 ***

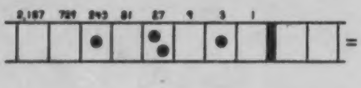
Put these numbers on the Base Three abacus.


2,187
729
243
81
27
9
3
1

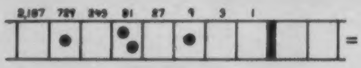
= 25


2,187
729
243
81
27
9
3
1

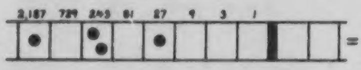
= 100


2,187
729
243
81
27
9
3
1

= 300


2,187
729
243
81
27
9
3
1

= 900

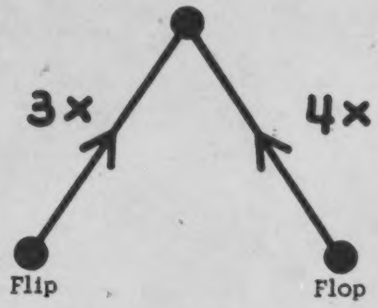

2,187
729
243
81
27
9
3
1

= 2,700

7.7 Fractions #3

Exercise 1

Draw this arrow picture on the board.



T: The three dots are for whole numbers. The arrow picture shows three times Flip equals four times Flop. What are some numbers that Flip and Flop could be?

We cannot predict how your students will respond. You should make a list on the board of numbers suggested by your students that Flip and Flop could be. The constraint of the three dots being for whole numbers can be lifted if the students wish to use decimals. In that case, we recommend that two charts be kept for numbers that Flip and Flop could be. One chart for whole numbers only and another for pairs with one of the numbers being a (non-whole number) decimal. The reason for this should become apparent as the exercise develops. In one of our development classes our charts looked like this:

Flip	Flop
4	3
12	9
8	6
20	15
60	45
40	30

Flip	Flop
2	1.5
$33\frac{1}{3}$	25
1	0.75
21	15.75

30 N

Point to the chart with only whole numbers.

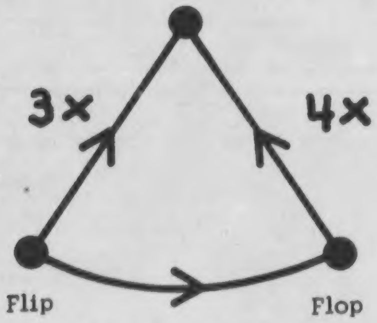
T: Look at this chart with only whole numbers, what do you notice about Flip?

S: Flip is a multiple of 4.

T: What about Flop?

S: Flop is a multiple of 3.

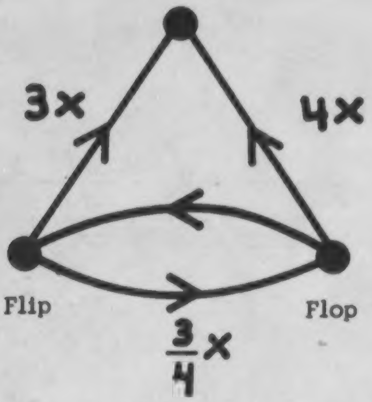
Add this arrow to your picture.



T: What could the blue arrow be for? The blue arrow is 3 x (trace the 3 x arrow) followed by "against" 4 x (trace against the 4 x arrow).

S: $\frac{3}{4}$ x.

Draw a return arrow from Flop to Flip in another color.



T: What could this arrow be for?

Trace the 4 x arrow followed by "against" the 3 x arrow for emphasis.

S: $\frac{4}{3}$ x.

Write the following problems on the board and ask your students to copy and to solve them. The correct answers are in the boxes.

$\frac{3}{4} \times 16 = \boxed{12}$ $\frac{4}{3} \times 30 = \boxed{40}$ $\frac{3}{4} \times 2 = \boxed{1.5}$

$\frac{3}{4} \times 40 = \boxed{30}$ $\frac{4}{3} \times 21 = \boxed{28}$ $\frac{3}{4} \times 10 = \boxed{7.5}$

$\frac{3}{4} \times 28 = \boxed{21}$ $\frac{4}{3} \times 75 = \boxed{100}$ $\frac{3}{4} \times 1 = \boxed{0.75}$

After the students have had a few minutes to work on these problems independently, check the problems collectively.

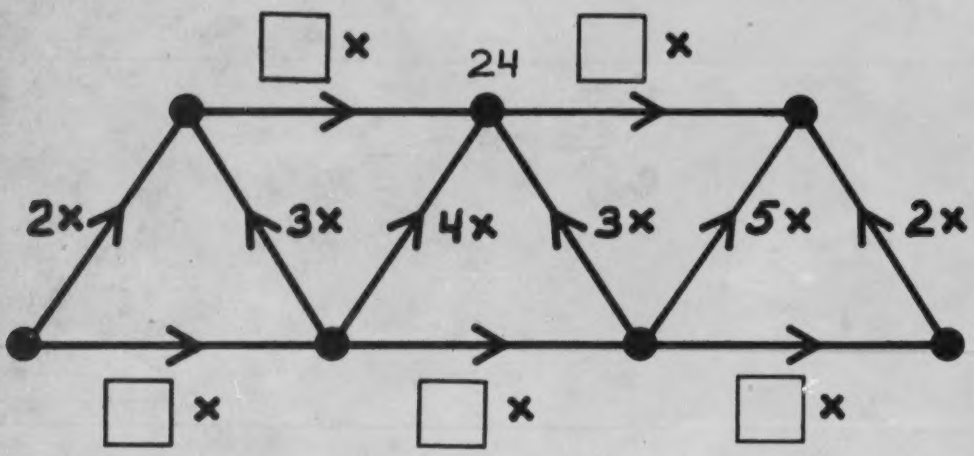
T: How did you calculate "$\frac{3}{4} \times 16$"?

S: $16 \div 4 = 4$ and $3 \times 4 = 12$, so $\frac{3}{4} \times 16 = 12$.

Continue in this manner for each of the remaining problems.

Exercise 2

Draw this arrow picture on the board.



T: We must label each dot and fill in the boxes for the blue arrow.

Invite students to label the dots, then the arrows. If a few students have difficulty labeling the arrows, start from the left side of the picture and trace the 2 x arrow. Then trace against the 3 x arrow as you say:

T: 2 x followed by "against" 3 x. What is another name for "against" 3 x?

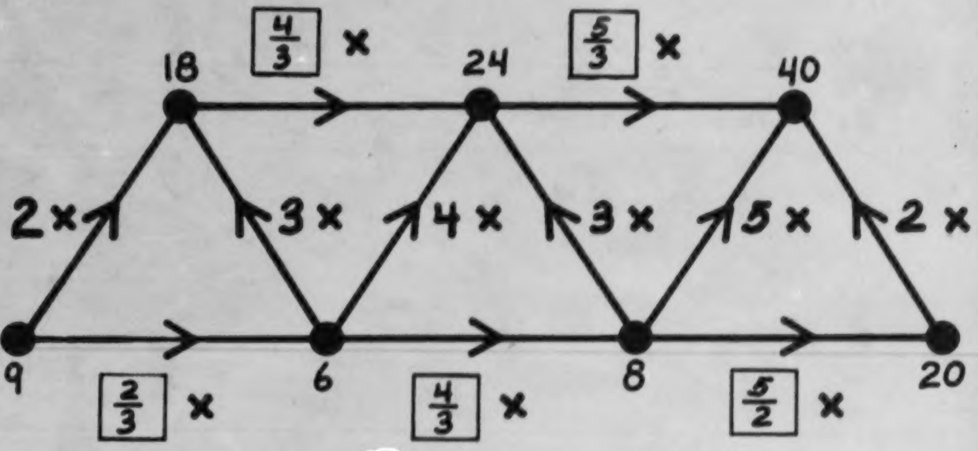
S: ÷ 3.

T: So 2 x followed by ÷ 3 is ... ?

S: $\frac{2}{3}$x.

331

Label the blue arrow "$\frac{2}{3}\times$" and continue in this manner, if necessary. A completed arrow picture is drawn below.

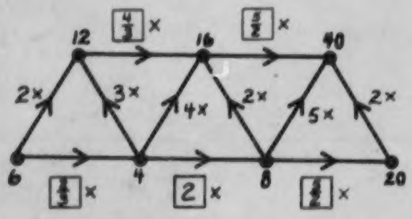


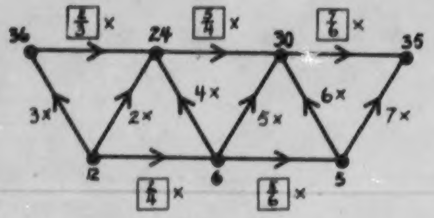
18
24
40
4/3 ×
5/3 ×
2×
3×
4×
3×
5×
2×
9
6
8
20
2/3 ×
4/3 ×
5/2 ×

Worksheets N72, N73, N74, and N75 are available.

Name ______________________ N 72 *

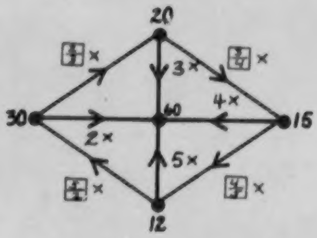
Label each dot. Fill in the boxes for the blue arrows.

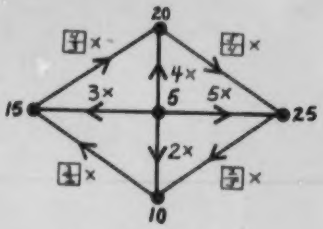

4/3 ×
5/2 ×
12
16
40
2×
3×
4×
2×
5×
2×
6
4
8
20
2/3 ×
2 ×
5/2 ×


2/3 ×
5/4 ×
7/6 ×
36
24
30
35
3×
2×
4×
5×
6×
7×
12
6
5
1/2 ×
5/6 ×

Name ______________________ N 73 **

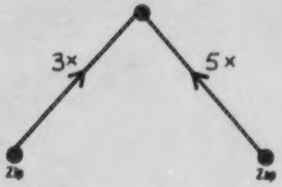
Label each dot. Fill in the boxes for the blue arrows.


20
3×
4×
60
30
15
2×
5×
12


20
4×
3×
6
5×
15
25
2×
10

Name ______________________ N 74 ***

First clue

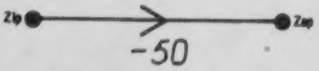


Complete this chart with pairs of numbers that Zip and Zap could be.
Many solutions are possible.

Zip	5	15	10	20	40	50	25	30	35	100	150	200	80	250
Zap	3	9	6	12	24	30	15	18	21	60	90	120	48	150

(Many solutions are possible.)

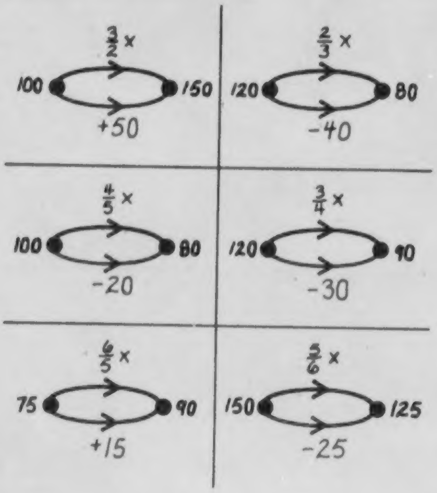
Second clue



Zip is 125. Zap is 75.

Name ______________________ N 75 ****

Label the dots.


3/2 ×
100
150
+50
2/3 ×
120
80
−40
4/5 ×
100
80
−20
3/4 ×
120
90
−30
6/5 ×
75
90
+15
5/6 ×
150
125
−25

8

DECIMAL DISTANCE

In this chapter, the concept of the "decimal distance" from one whole number to another is introduced in a story situation involving two bus lines in a mythical city that make trips from house to house at a penny a stop. A red bus goes from one house to another if and only if the number of one of the houses is one more than the number of the other; a blue bus goes from one house to another if and only if the number of one of the houses is ten times the number of the other house. Trips are represented by a chain of cords;

+1 or -1 ×10 or ÷10

The decimal distance from one number to another is the least number of cords in a trip (a chain of cords) from one number to the other; that is, the least cost from one house to another at a penny a cord. After some experiences in determining shortest routes between given numbers, a detective story is posed concerning the house number of one of Nabu's friends, with the clues involving decimal distances from Nabu's house and from another friend's house. Other activities center on problems of finding all numbers at a given decimal distance from a given number and finding numbers at the same decimal distance from two given numbers.

CAPSULE LESSON SUMMARIES

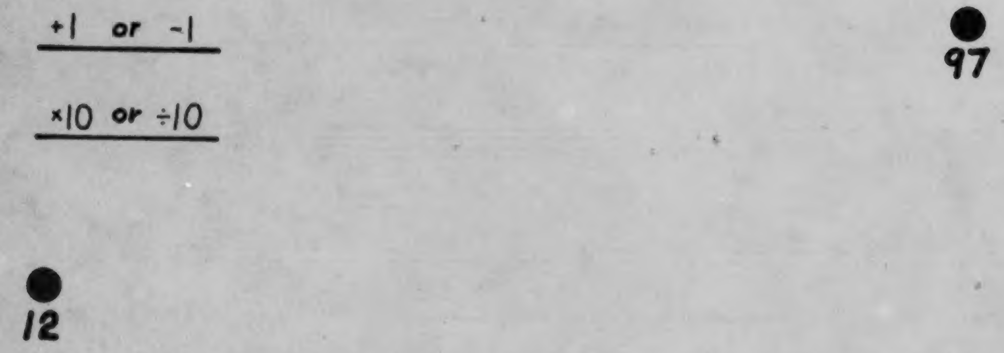
Section 8.1:	Using red and/or blue cords, find a shortest chain of cords joining two given whole numbers, where the numbers are joined by a red cord if and only if one of them is one more than the other, and the numbers are joined by a blue cord if and only if one of them is ten times the other.
Section 8.2:	For a given whole number, find all the whole numbers at a specified decimal distance from it, where the decimal distance from one whole number to another is the number of cords in a shortest chain of cords between them. Given two whole numbers, find numbers that are at the same decimal distance from them.

12 N

8.1 Decimal Distance #1

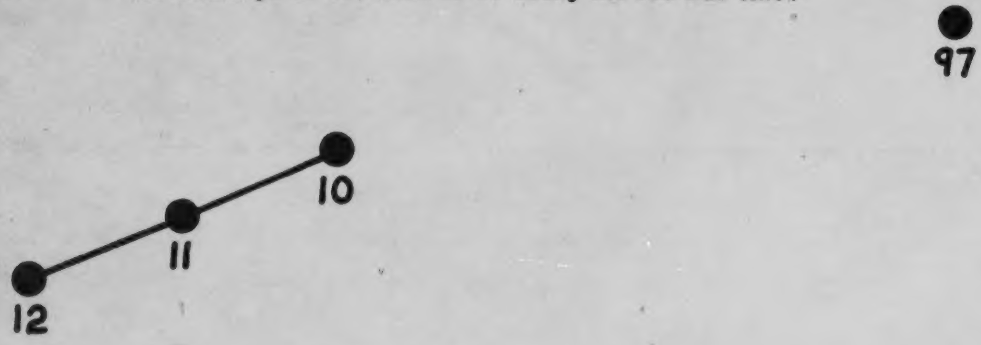
Exercise 1

Draw this picture on the board.



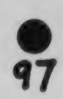
T: Nabu lives in a town that has exactly two bus lines -- red and blue. The bus lines have a strange way of calculating one's fare: the passenger must pay a penny for each house that is passed on a trip. The citizens of Nabu's town have made a game out of riding the bus. Whenever they ride the buses, they try to find the shortest routes to their destinations. How could Nabu travel from house number 12 to house number 97, spending as little money as possible? By the way, all the houses have WHOLE number addresses.

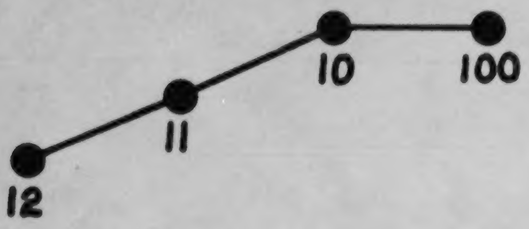
S: He could go to 11, then to 10 using the red bus line.



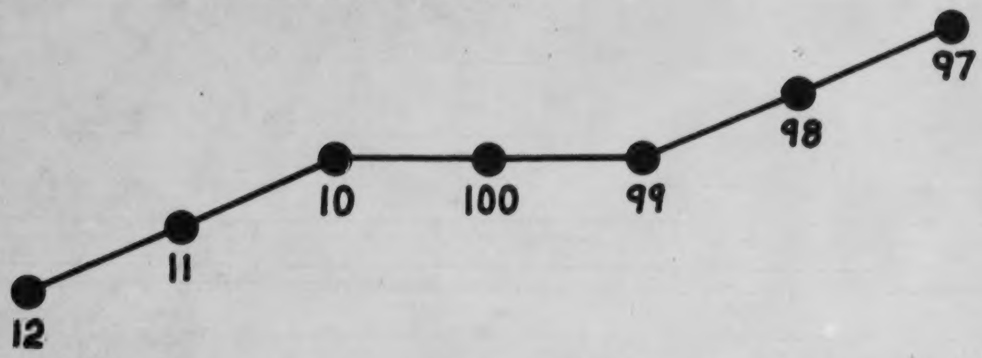
T: Should he use red or blue next?

S: Blue. He could go to 100.





T: Who can complete the picture?



This is the shortest route from 12 to 97; your students might suggest a longer route in which case you should direct the discussion until this route is discovered.

T: How much would this trip cost Nabu?

S: Six cents.

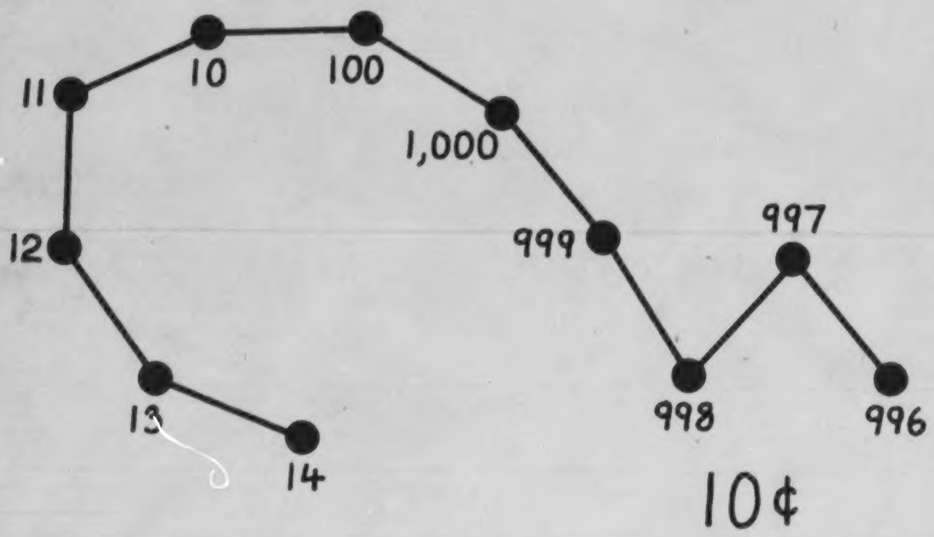
T: On your paper, show a trip from 14 to 996.

Let the students have a few minutes to work on the problem. After a while, ask a student to begin putting a solution on the board.

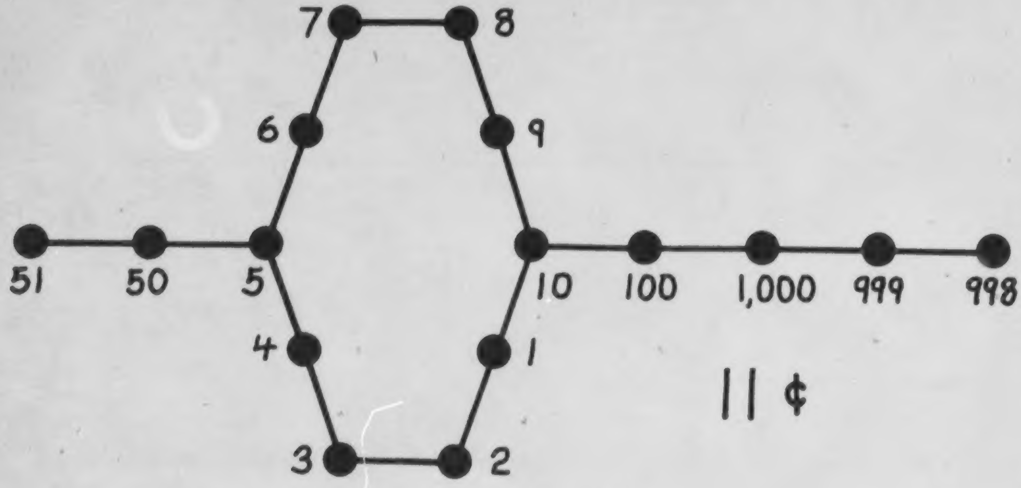
12 N

Let the class work slowly and progressively presenting a solution, allowing those students who have not yet found a solution an opportunity to complete the problem with hints, but on their own.

A route between 14 and 996 is shown below; this is the shortest one and should eventually be presented by the class.


11
10
100
1,000
12
999
997
13
14
998
996
10¢

Repeat the above activity using 51 and 998. Be sure to solve the problem collectively as above. For this problem, there are two shortest routes, both of which are shown below.


7
8
6
9
51
50
5
10
100
1,000
999
998
4
1
11 ¢
3
2

337

At this point, write the four problems below on the board and let the students work on them independently. After several minutes, ask students who have solutions to put them on the board. Solutions are given in the following illustrations.

Draw a shortest route between:

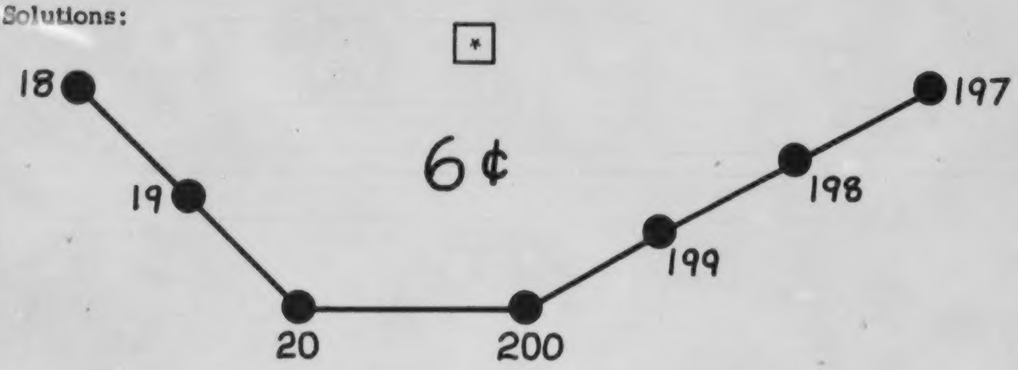
*
18 and 197

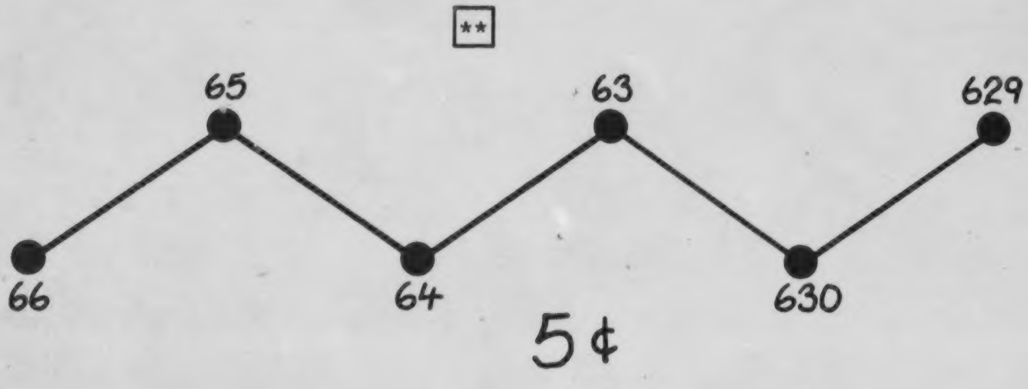
**
66 and 629

***
44 and 996

****
777 and 888

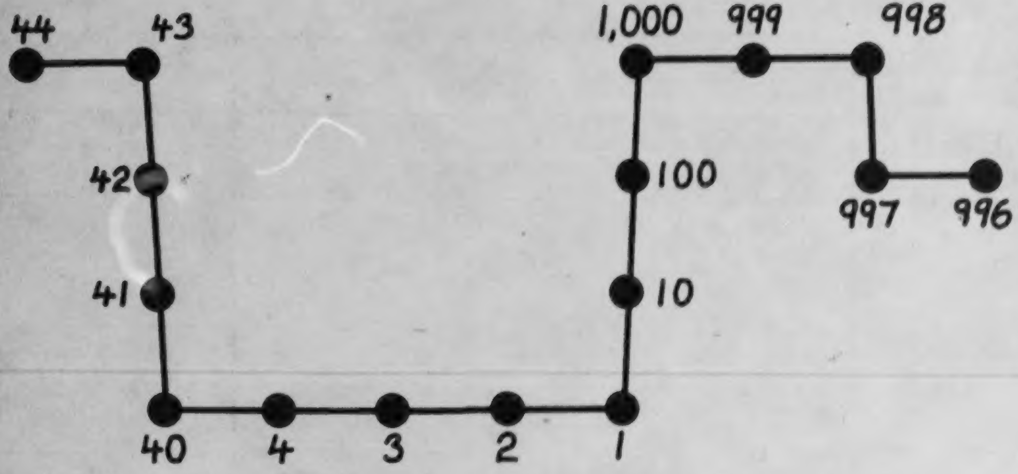
Solutions:


*
18
19
20
200
199
198
197
6¢


**
66
65
64
63
630
629
5¢

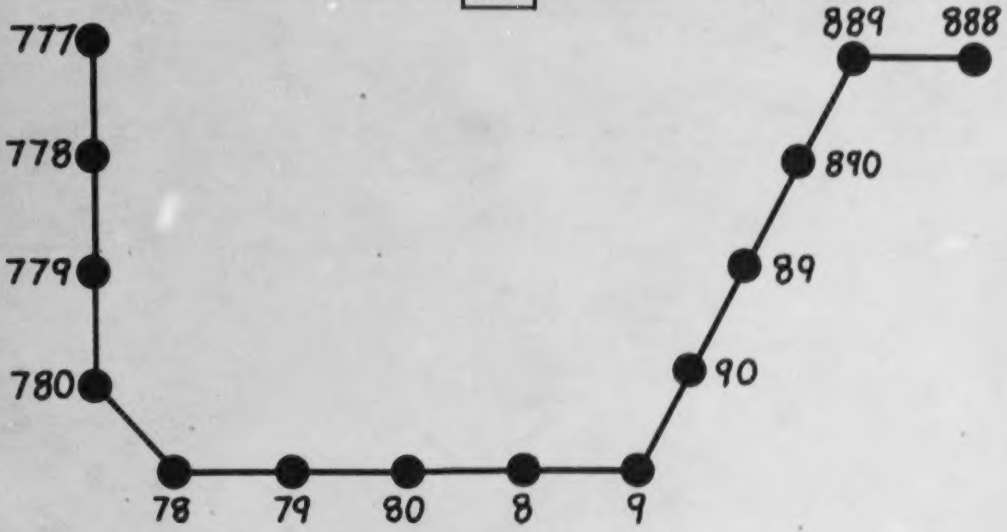
12 N

***


44
43
42
41
40
4
3
2
1
10
100
1,000
999
998
997
996

15¢

****



13¢

Exercise 2

T: Nabu has a friend Nick. This is a detective story about Nick's house number.

First Clue

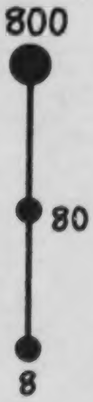
T: Nabu's house number is 8. Nabu must spend 2¢ to visit his friend Nick.* Where could Nick live?

S: Nick could live in number 800.

T: How would Nabu go to Nick's house.

S: He would go to 80, then to 800 using two blue cords.

Draw this picture on the board.



NOTE: Big dots are used only to emphasize the possible numbers for Nick's house.

---

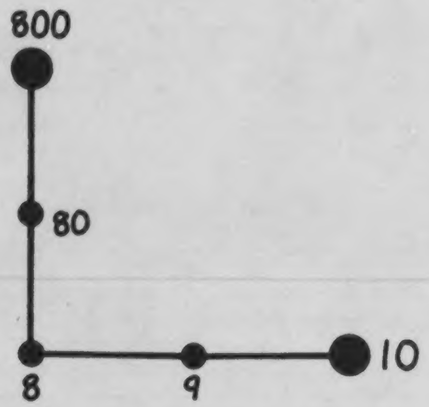
* Nabu and Nick do not live in the same house, thus ruling out a trip from 8 to 8 for 2¢.

12 N

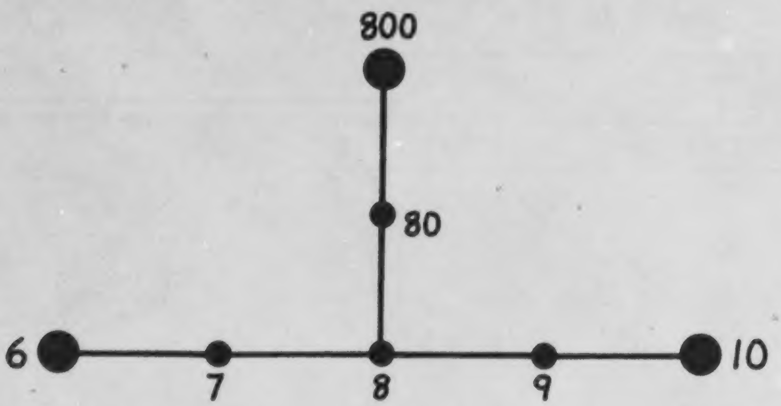
T: Where else could Nick live?

S: He could live at number 10 -- two red cords.

Add this road to your drawing.



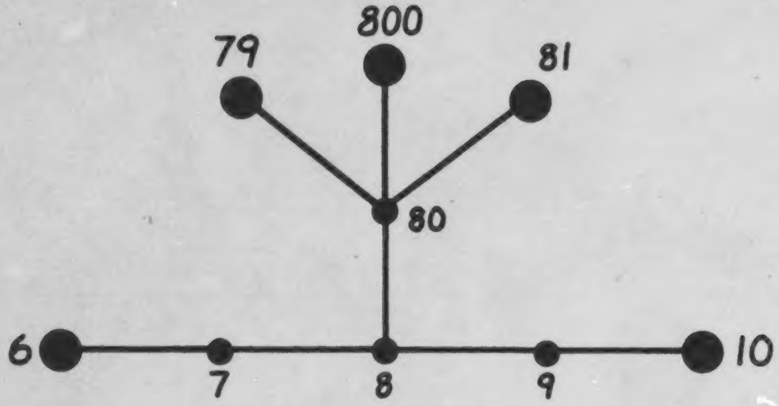
S: He could live at number 6 -- again, two red cords.



T: Are these three numbers the only ones possible for Nick's house?

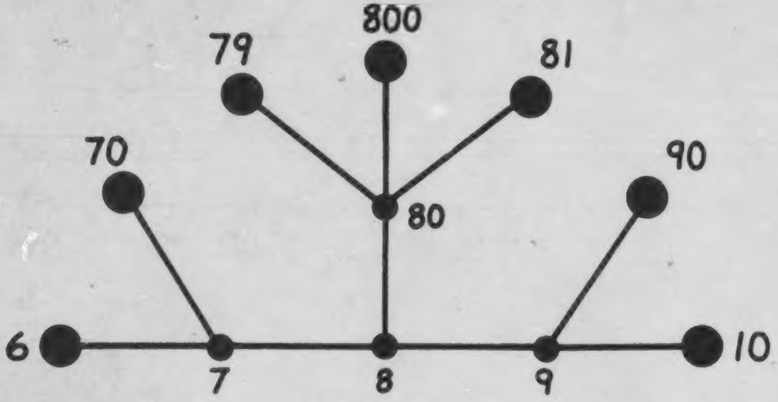
S: No. Nabu could go to 80 then take a red cord to 81 or 79.

Add this information to your picture.


800
79
81
80
6
7
8
9
10

T: Are there any other possibilities?

The class should find the two remaining roads. Be sure that each road is added to your picture.


800
79
81
70
90
80
6
7
8
9
10

The class should conclude that Nick lives at 6 or 10 or 70 or 79 or 81 or 90 or 800.

Second clue

T: Nick has a friend, Nack, who lives at number 999. Nick can visit Nack and only spend 3¢. Can this clue help you to discover Nick's house number?

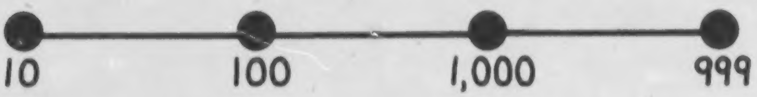
S: Yes, but we must build some roads.

T: Can you explain?

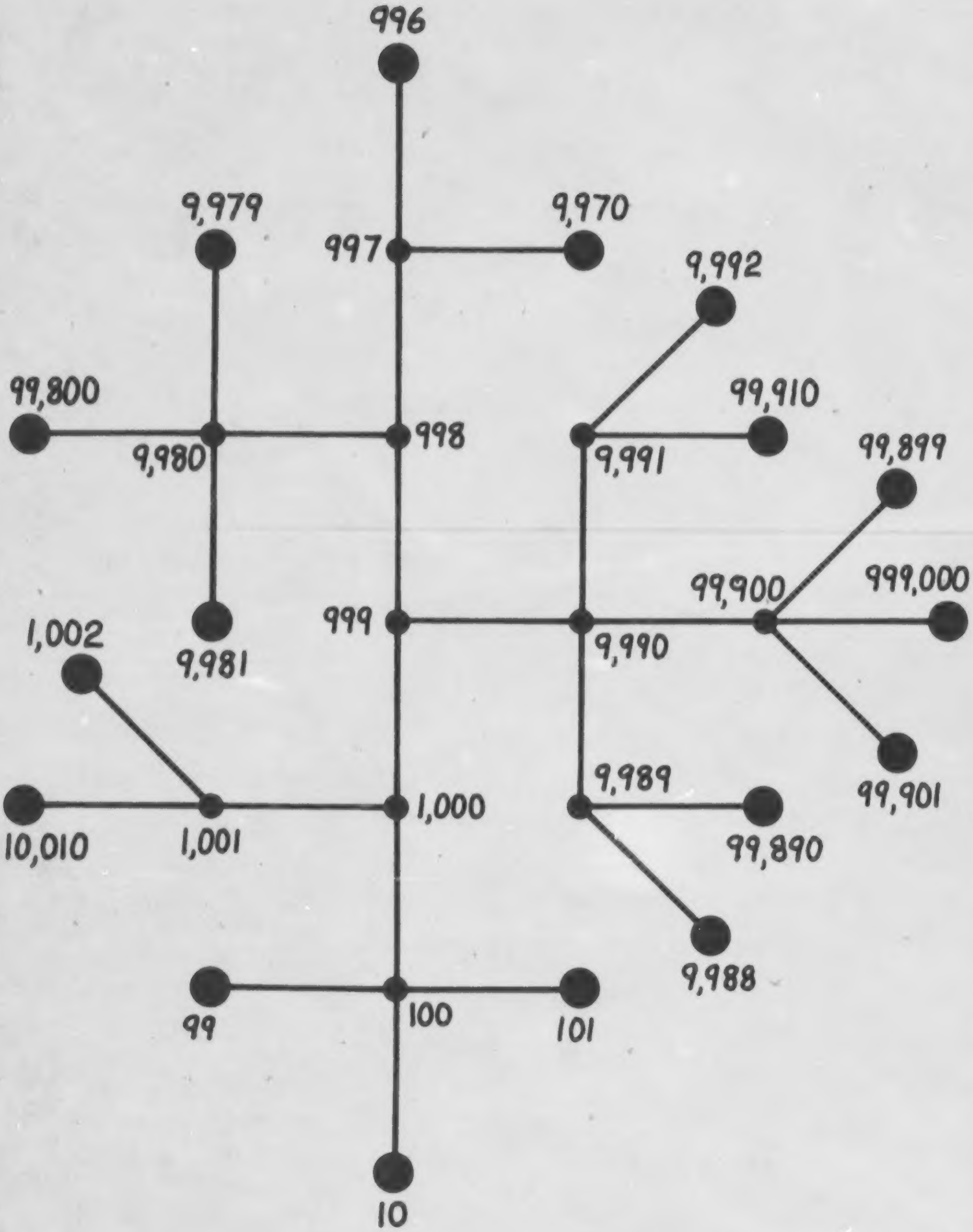
S: We must try to build a road between 6 and 999, for example, that would cost 3¢. Then we do the same for 10 and 999; for 70 and 999; and so on.

Once it is clear to the students that there are seven road problems to be tested, let them work independently. If you find that you are pressed for time, the work could be divided among the class; for example, the first row could try to draw a 3¢ road between 6 and 999; the second row could try to draw a 3¢ road between 10 and 999; etc.

When it is discovered that Nick must live at number 10, ask a student to present the 3¢ road between 10 and 999 and discuss why you can eliminate the other possibilities.



The following picture is only for your information; you should not expect your class to draw it. It shows all the numbers that are a (decimal) distance 3 or less from 999 and since of the possibilities for Nick's house number, only 10 appears in this picture, we are sure that Nick lives at number 10.

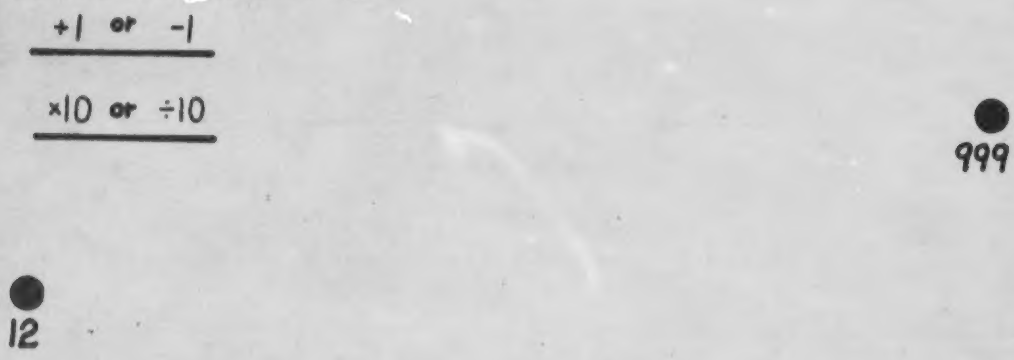

996
9,979
997
9,970
9,992
99,800
9,980
998
9,991
99,910
99,899
99,900
999,000
1,002
999
9,990
9,981
99,901
9,989
10,010
1,001
1,000
99,890
9,988
99
100
101
10

18 N

8.2 Decimal Distance #2

Exercise 1

Draw this picture on the board.



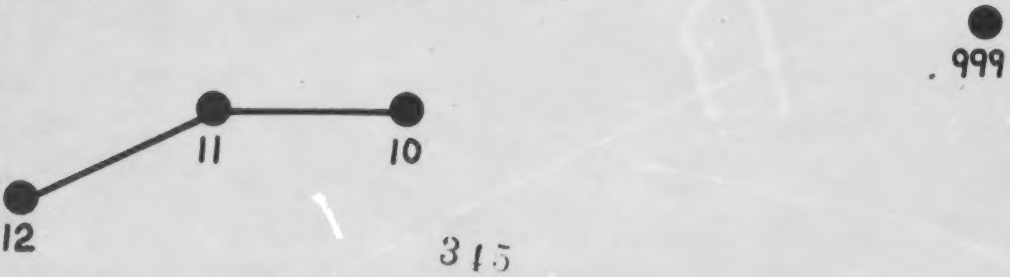
T: Do you remember the story of Nabu and the red and blue bus lines?

Let the students recall as must of the story as they can. Three points from the previous lesson should be recalled:

- The town has exactly two bus lines -- red and blue.
- A passenger must pay 1¢ for each house that is passed on a trip.
- Passengers try to find shortest routes to their destinations.

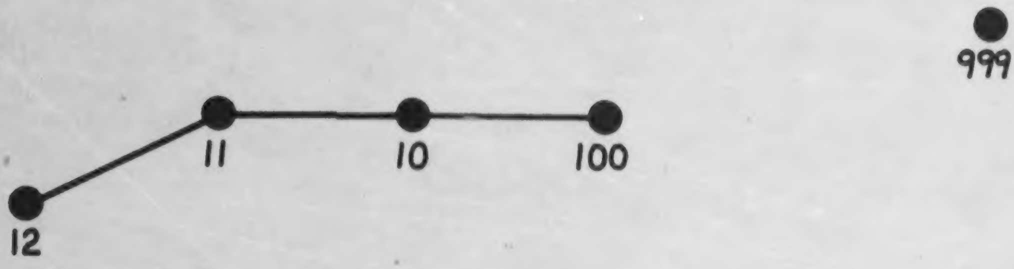
T: How could Nabu travel from house number 12 to house number 999, spending as little money as possible?

S: He could go from 12 to 10 using two red cords.

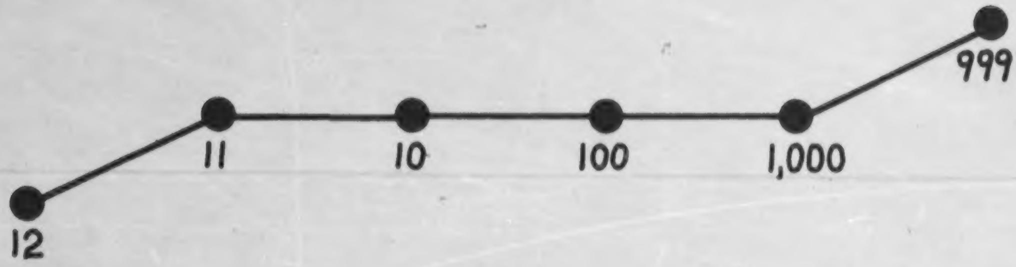


T: How should he go on, with a red or blue cord?

S: Blue to 100.



T: Who can complete the picture?



This is the shortest road from 12 to 999; direct the discussion so that the class discovers this road.

T: How much did the trip cost?

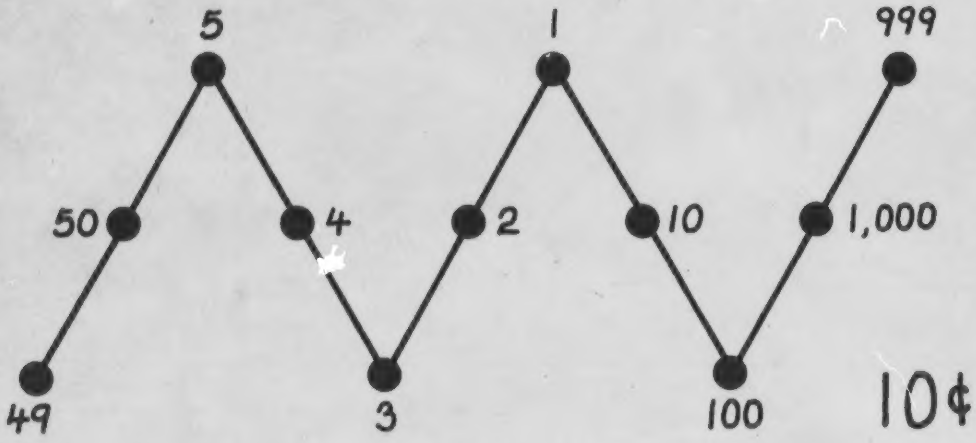
S: Five cents.

T: On your paper, draw a trip from 49 to 999. Try to spend as little money as possible.

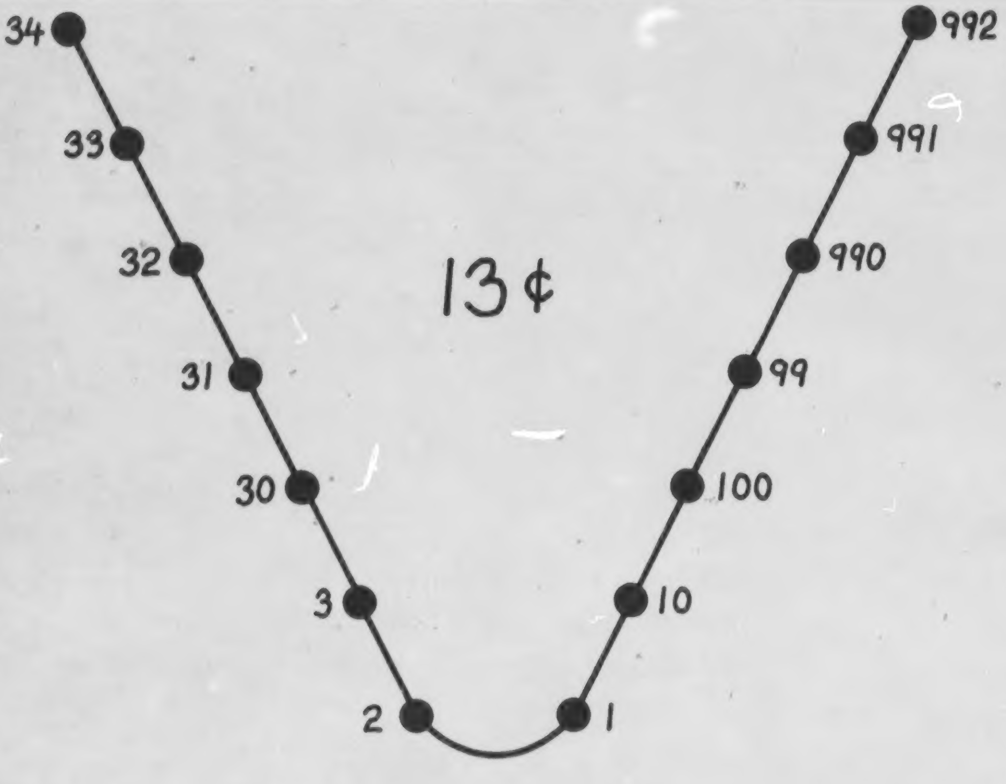
Let the students have a few minutes to consider the problem. After a while, ask a student to begin putting a solution on the board. The solution should be presented slowly and progressively, allowing those students who have not yet found a solution an opportunity to complete the problem.

18 N

A shortest road between 49 and 999 is shown below.


5
1
999
50
4
2
10
1,000
49
3
100
10¢

Repeat the above activity using 34 and 992. Be sure to solve the problem collectively as above. The shortest road between 34 and 992 has 13 cords.

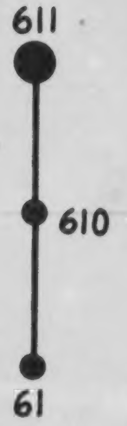

34
992
33
991
32
990
13¢
31
99
30
100
3
10
2
1

Exercise 2

T: Nabu was at house number 61 and had only 2¢ to spend for a bus ride. Which houses could he visit?

S: He could visit house number 611 -- a blue cord followed by a red cord.

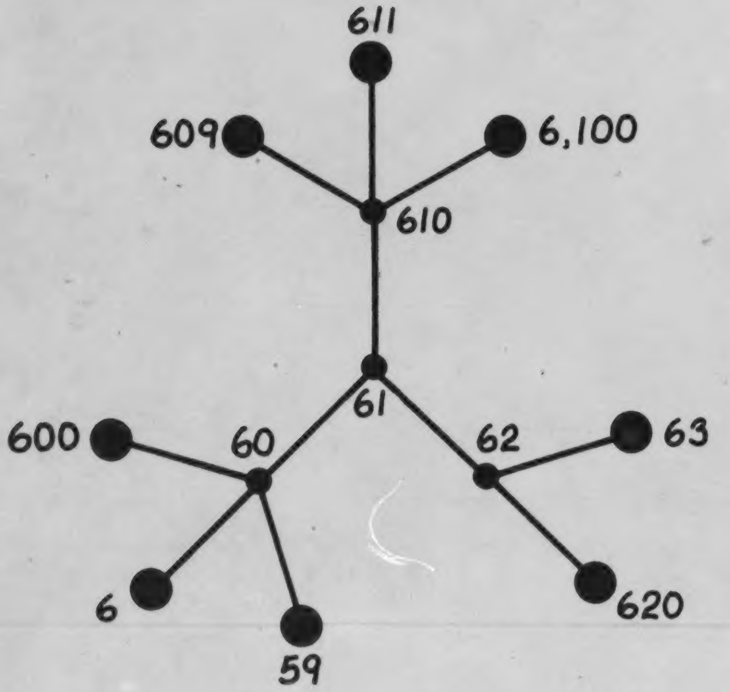
Draw this picture on the board.



NOTE: Big dots are used only to emphasize the numbers of houses that Nabu could visit.

T: Try to find all the houses Nabu could visit for 2¢.

Let the students work independently on this problem for a few minutes; then complete the problem collectively. A completed picture is shown on the following page.

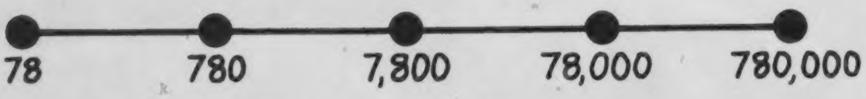

611
609
6,100
610
61
600
60
62
63
6
59
620

Exercise 3

T: Nabu is at house number 78 and has exactly 4¢ to spend for a bus ride. Which is the largest house number where he could visit?

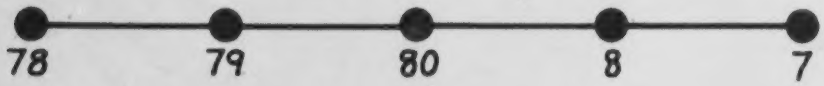
S: Use four blue cords.

Draw the road and ask students to label the dots.

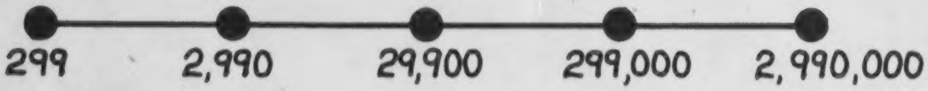

78
780
7,800
78,000
780,000

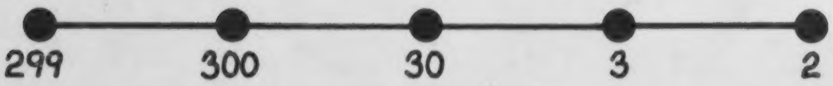
T: 780,000 is the largest. If Nabu starts at 78, which is the smallest house number where he could visit for exactly four cents?

It is probable that some students will suggest that 74 is the smallest (78 to 74 with four red cords). However, this is not the smallest. The road to the smallest number, 7, is shown below.



Repeat the above activity using 299 and again 4¢ to spend. The roads to the largest and smallest numbers are shown below.





Exercise 4

T: Nick lives in house number 9 and Nack lives in house number 1,001. They plan a secret meeting in another house. To get to the house by bus, each of them had to pay exactly the same price. In which house could the secret meeting take place?

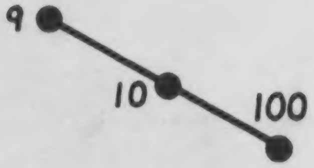
Let the students have a few minutes to consider this situation; then discuss the problem collectively.

S: They could meet at 100.

T: How could we draw a road from 9 to 100?

18 N

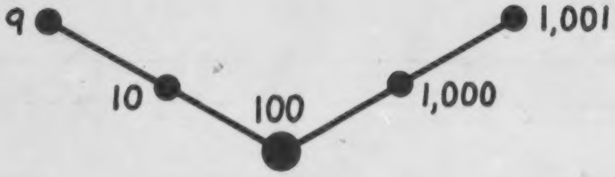
S: Red then blue.



T: Can we go from 1,001 to 100 for 2¢?

S: Yes, red then blue.

Draw this road.

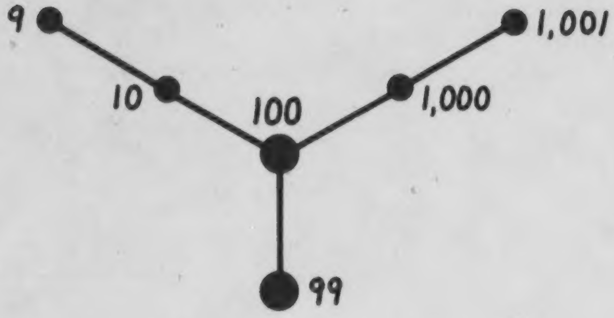


T: Good. Nick and Nack could meet at 100 and each would have to pay two cents. Are there any other houses where they could meet?

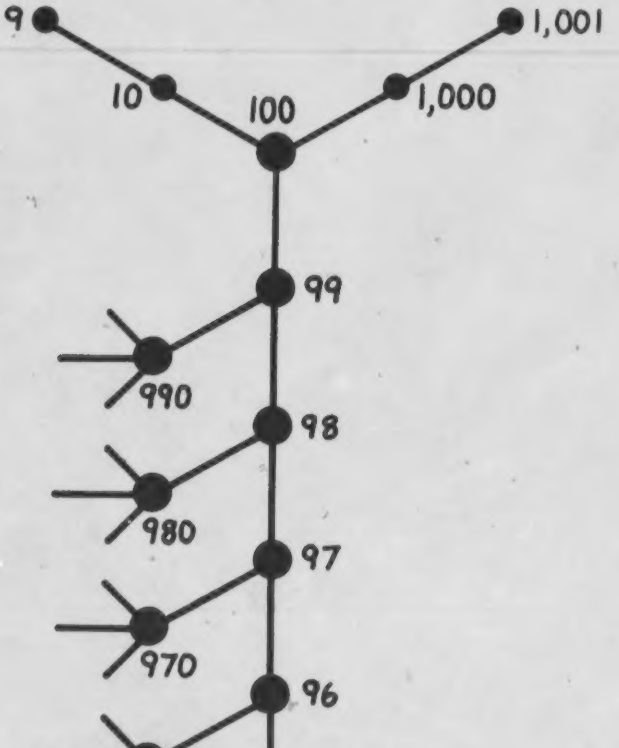
S: They could meet at 99. First go to 100, then to 99.

T: Good. Both Nick and Nack would have to pay 3¢.

Add this red cord to your picture.

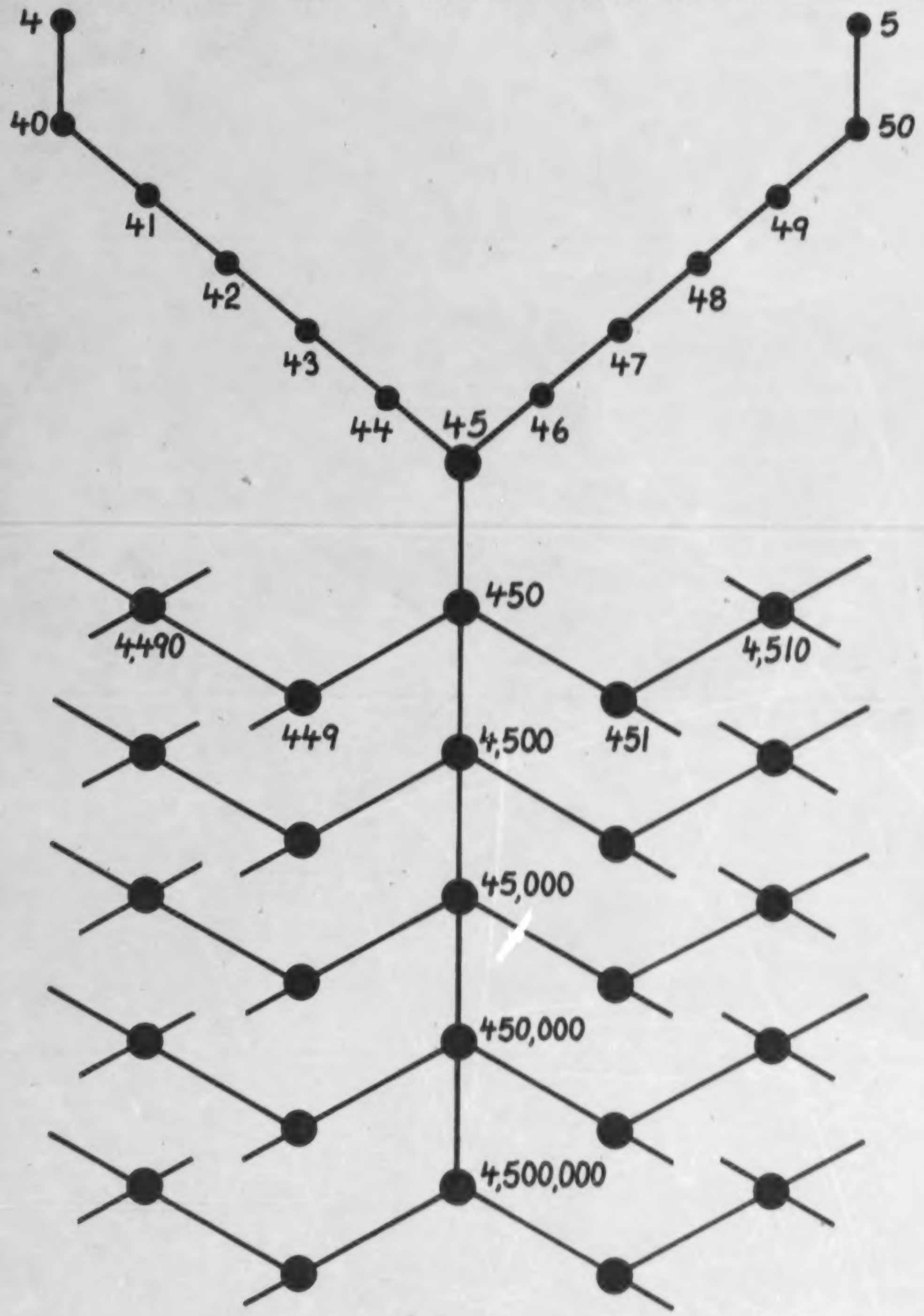

9
10
100
1,000
1,001
99

Continue expanding your picture in this manner. There are many places where the boys could meet. A partially complete picture is drawn below.


9
10
100
1,000
1,001
99
990
98
980
97
970
96
960

18 N

Repeat the preceding activity with 4 and 5 as the starting points. Again, there are many possible meeting places. A partially complete picture is drawn below.


4
5
40
50
41
49
42
48
43
47
44
45
46
450
4,490
4,510
449
4,500
451
45,000
450,000
4,500,000

9

DIVISION

The lessons in this chapter reinforce and extend the experiences of Parts I and II of the CSMP Intermediate Grades Program relative to situations involving division. Arrow roads continue to be the primary vehicle for preparing students for the division algorithm. Composites of functions of the form "÷■" are introduced for the first time, where the "■" can be any positive integer.

CAPSULE LESSON SUMMARIES

Section 9.1: Solve division problems suggested by situations in which a given number of bottles are packed in boxes, each box holding a certain number of bottles. Build arrow roads and use the results to solve the division problems.

Section 9.2: Continue the activities of Section 9.1 with a greater variety of situations suggesting division.

Section 9.3: Determine which numbers could be labels for the dots in an arrow picture and then identify arrows of the form "÷■", where "■" is any positive integer. Label certain composite arrows in the arrow picture. (Worksheets N76, N77, N78, N79)

16 N

9.1 Division #1

Exercise 1

Draw this table on the board.

Number of Bottles	Number of Bottles in Each Box	Number of Boxes Completely Filled	Number of Bottles Left Over
27	6		

Point to the appropriate places in the table as you say:

T: I would like to put 27 bottles into some boxes. Each box holds 6 bottles. How many boxes will I fill completely and how many bottles will be left over?

S: 4 boxes will be filled, because 4 × 6 = 24.

S: 3 bottles will be left over.

Add this information to your table and write the calculations (both of these number stories) on the board.

$$(4 \times 6) + 3 = 27 \qquad \begin{array}{r} 4 \\ 6\overline{)27} \end{array} \; R = 3$$

Number of Bottles	Number of Bottles in Each Box	Number of Boxes Completely Filled	Number of Bottles Left Over
27	6	4	3

Continue in the same way for the information in the table below. Correct answers are in the boxes; students should provide these entries.

Number of Bottles	Number of Bottles in Each Box	Number of Boxes Completely Filled	Number of Bottles Left Over
27	6	4	3
76	10	$\boxed{7}$	$\boxed{6}$
235	10	$\boxed{23}$	$\boxed{5}$
688	20	$\boxed{34}$	$\boxed{8}$
4,535	100	$\boxed{45}$	$\boxed{35}$

Exercise 2

Put this information on the board.

$\xrightarrow{+250}$ $\xrightarrow{+25}$ $\xrightarrow{+\square}$ ● 1,639

The black arrow should be drawn with white (or your regular) chalk.

0 ●

16 N

T: The white arrow will be for plus a whole number smaller than 25. Later we will see what it could be.

Let's build a red-blue-white road from 0 to 1,639 using as few arrows as possible. Can you predict the number of blue arrows we should draw?

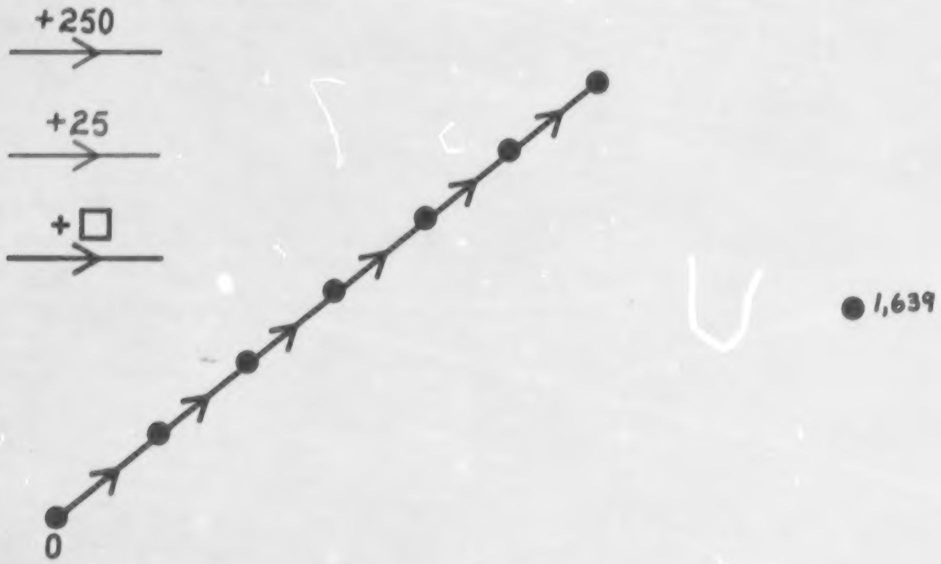
If necessary, remind the students that the road is to be built using as many blue arrows as possible, then as many red arrows as possible, and then the white arrow.

S: 6, because 6 × 250 = 1,500 and 7 × 250 = 1,750, which is too large.

Students should have several minutes to work independently on this problem; then build the road collectively.

T: You said we should use 6 blue arrows.

Draw this part of the arrow road.



T (pointing to the last unlabeled dot): Which number is here? Why?

S: 1,500, because 6 × 250 = 1,500.

Label the last dot "1,500". If you find it necessary, the class could label all the unlabeled dots.

T: Could we draw another blue arrow?

S: No, 7 × 250 = 1,750 and that's too big.

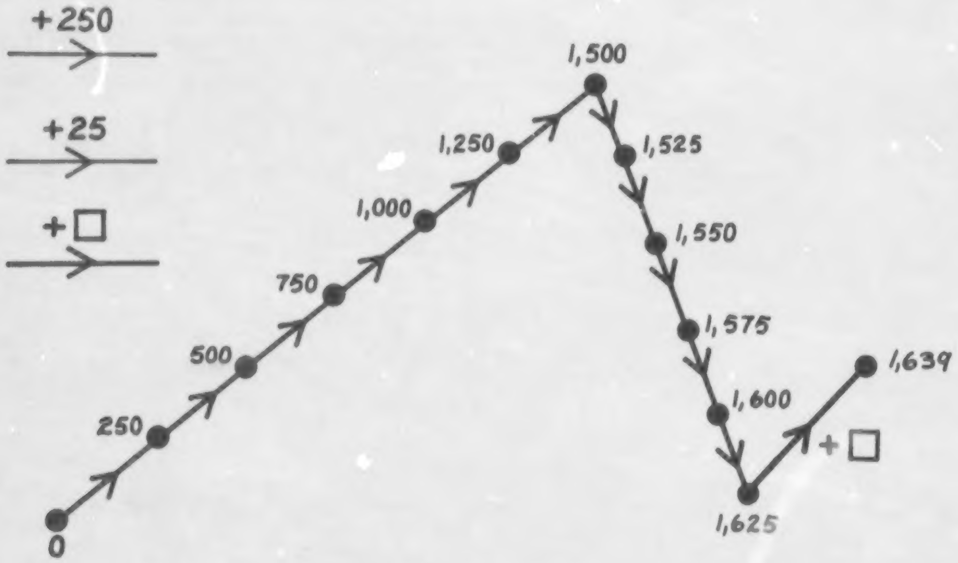
T: How many red arrows should we draw? Why?

S: 5 red arrows; 5 × 25 = 125 and 1,500 + 125 = 1,625.

T: Could we use more than 5 red arrows?

S: No, 6 × 25 = 150 and 1,500 + 150 = 1,650, which is bigger than 1,639.

Add the five red arrows to your picture and ask the students to label at least the last dot of the red road. All the dots have been labeled in the picture below.



T: What is this white arrow (shown in black above) for?

S: +14.

Label the arrow "+14".

T: Our friend, Bruce, drew this same arrow road. However, he was using the road to solve a problem. What problem do you think Bruce was trying to solve?

Let the students express their opinions of this situation. For each suggestion, encourage the student to explain how the arrow picture could be used to solve the problem. For example,

S: Bruce could be putting bottles into boxes of 250 and into boxes of 25.

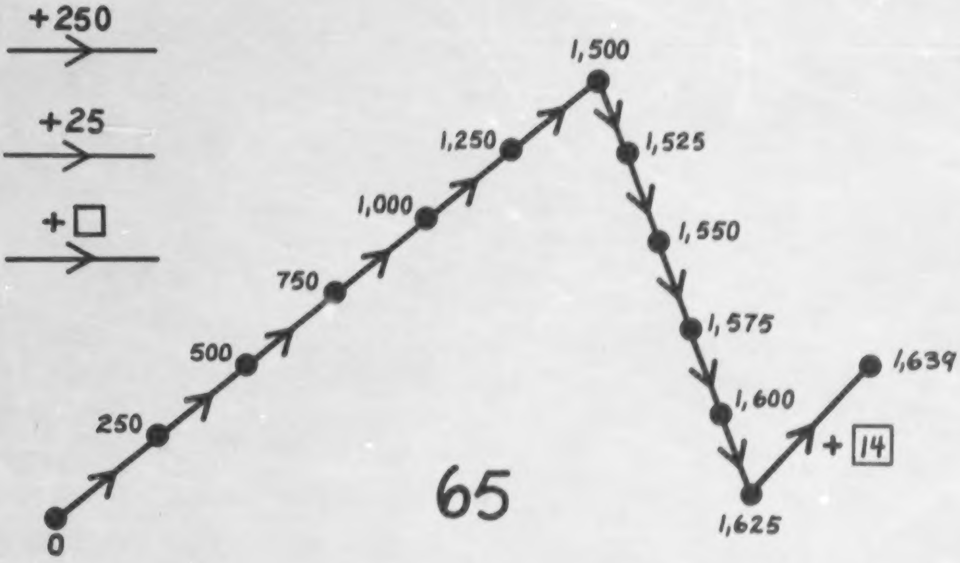
T: How would Bruce use the arrow picture to help solve this problem?

S: If Bruce has 1,639 bottles, the arrow picture shows filling 6 boxes of 250 (6 blue arrows), and 5 boxes of 25 (5 red arrows), and having 14 bottles left over. Each blue arrow would be for a box of 250, and each red arrow for a box of 25, and the white arrow is for the left over.

S: Bruce might be filling boxes of 25 bottles each. Then each blue arrow could be for filling 10 boxes of 25 bottles each. Each red arrow could be for filling 1 box of 25 bottles each.

If the above situation is not mentioned by someone, mention it yourself.

Write "65" with "6" in blue and "5" in red near the arrow road as shown below.



NOTE: Try to arrange your board space so that each sub-road continues in one direction (and different sub-roads in different directions). This arrangement highlights the link between the arrow road and the red and blue numerals.

T: When Bruce solved his problem, he wrote this (point to "65"). Why do you think he wrote this?

Let the students express their opinions of this situation.

S : The blue "6" could be for 60 boxes of 25 bottles each, and the red "5" could be for 5 boxes of 25 bottles each. 65 boxes of 25 bottles each.

T: Here is another way to express this problem of putting bottles into boxes.

Write this calculation on the board.

$$\begin{array}{r} 65 \\ 25\overline{)1{,}639} \end{array} \quad R = 14$$

Exercise 3

Write this problem on the board.

$$15\overline{)1{,}153}$$

T: Let's use an arrow road to help solve this problem. We must build an arrow road from 0 to which ending number?

S: 1,153.

Draw two dots on the board and label them "0" and "1,153", respectively.

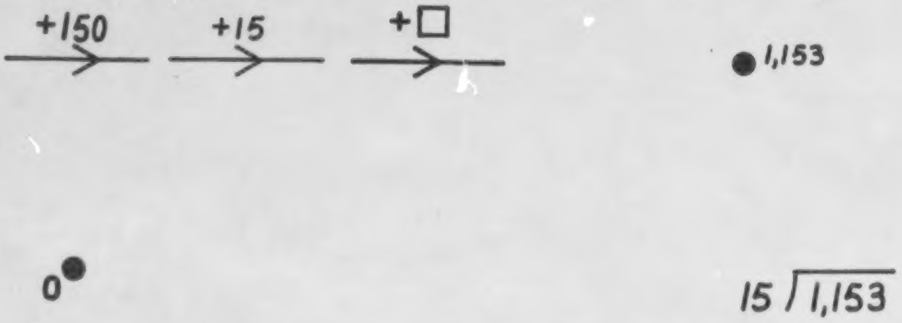
T: What kind of arrows could we use?

S: + 150. A + 150 arrow would be for filling 10 boxes of 15 bottles each.

S: + 15. A + 15 arrow would be for filling 1 box of 15 bottles each.

S: We may need a "+☐" arrow for the left over bottles.

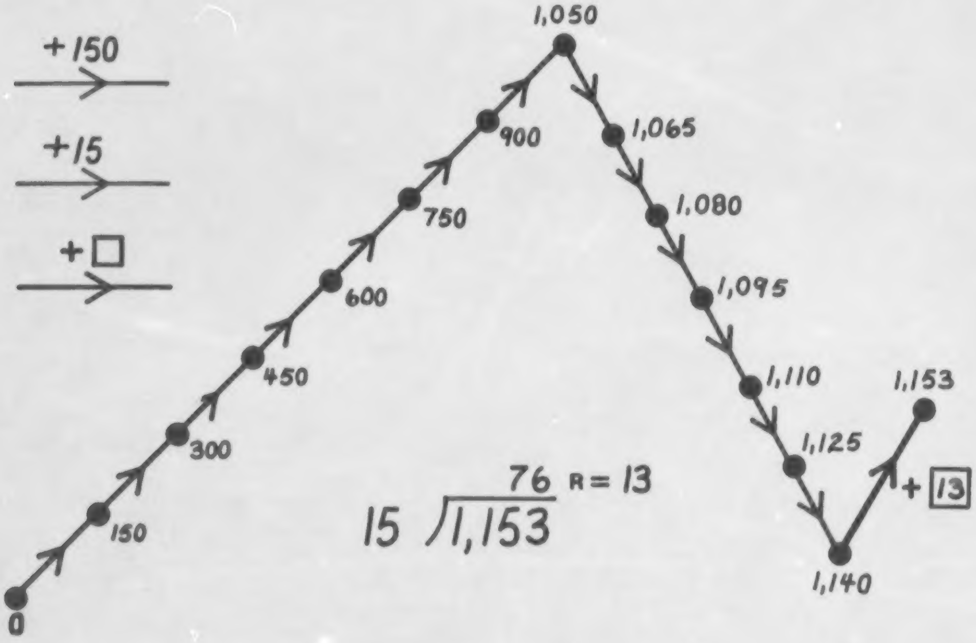
Draw these three arrows on the board using three different colors.


+150
+15
+□
1,153
0
15)1,153

NOTE: Emphasize when you write "+□" for the third arrow that the number in the box should be less than 15.

Let the students work on this problem independently for several minutes. Observe the students' progress and help those who have difficulty getting started.

When it is obvious that most students have finished the problem, check it collectively in a manner similar to Exercise 2. A completed arrow road is shown below.


+150
+15
+□
0
150
300
450
600
750
900
1,050
1,065
1,080
1,095
1,110
1,125
1,140
+13
1,153
76 R = 13
15)1,153

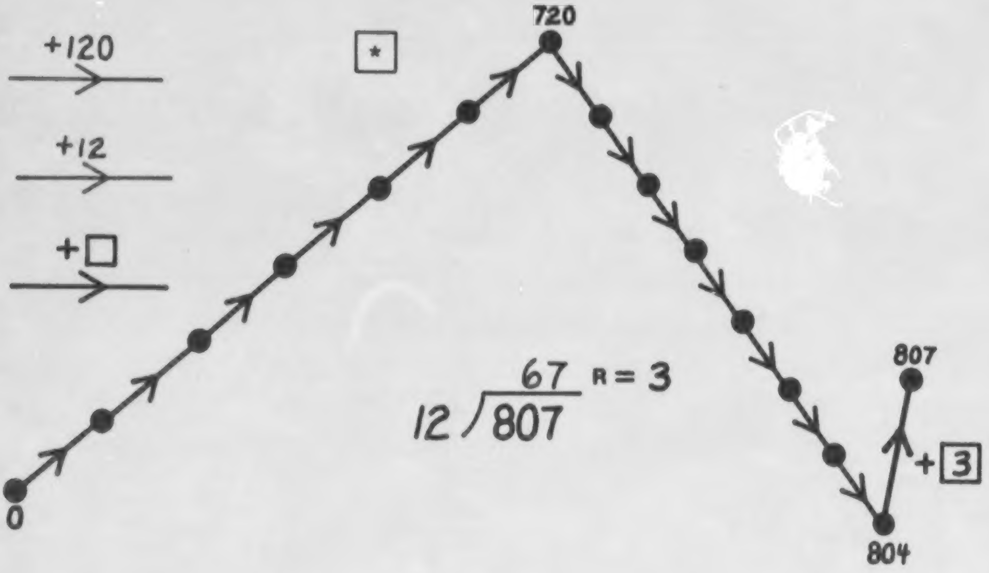
16 N

Write these three problems on the board and ask the students to solve them. Suggest they use arrow roads to help. Complete solutions are given below.

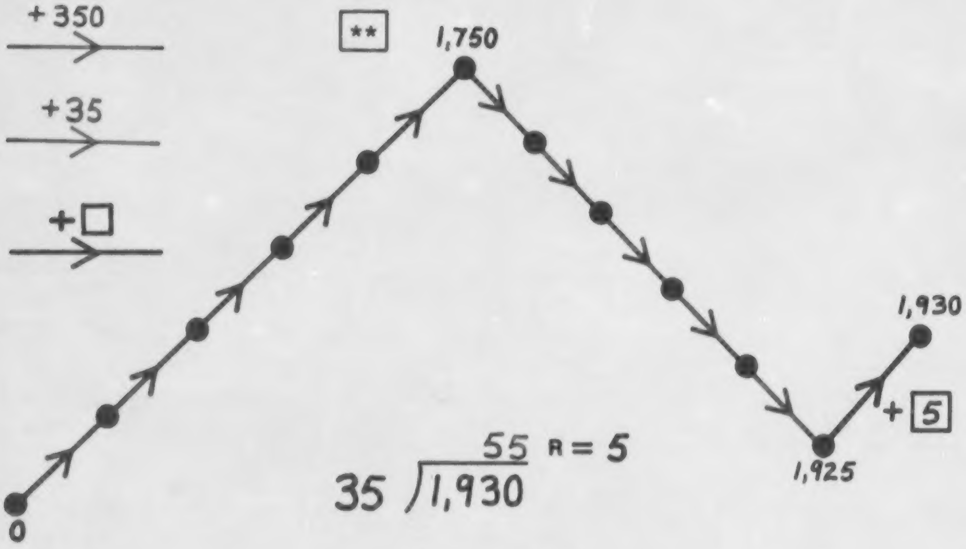
$\boxed{*}$ $12\overline{)807}$

$\boxed{**}$ $35\overline{)1{,}930}$

$\boxed{***}$ $75\overline{)25{,}890}$

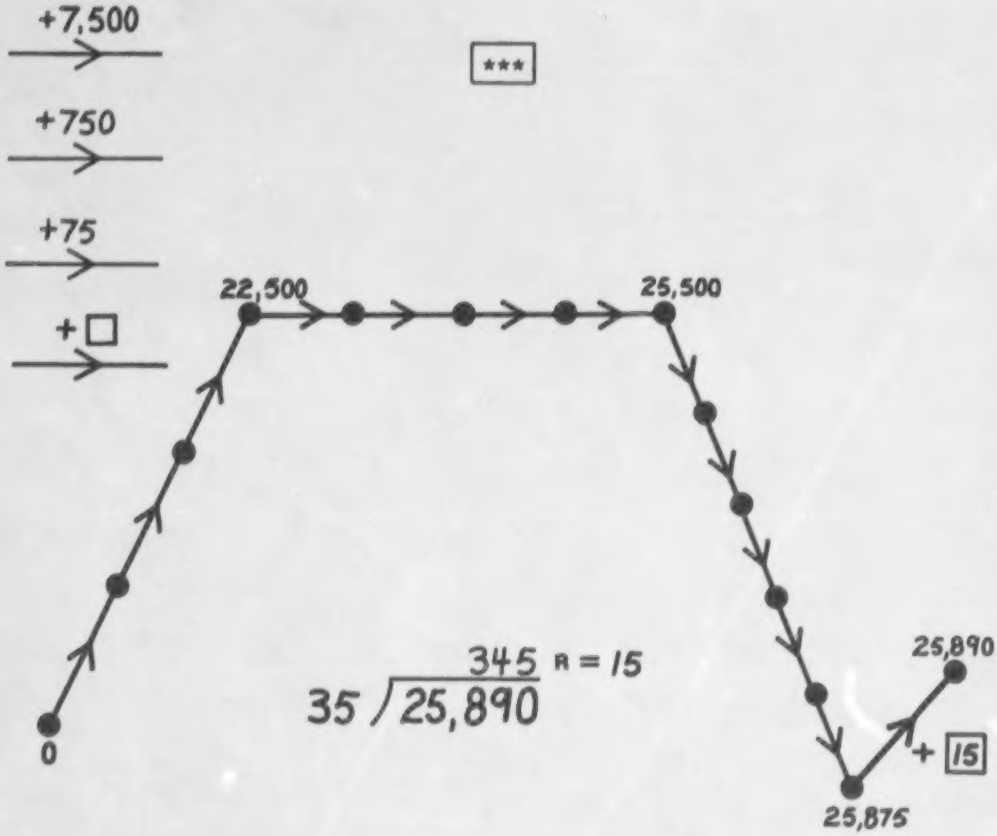

+120
+12
+□
*
720
807
+3
0
804

$\begin{array}{r} 67 \\ 12\overline{)807} \end{array}$ R = 3


+350
+35
+□
**
1,750
1,930
+5
1,925
0

$\begin{array}{r} 55 \\ 35\overline{)1{,}930} \end{array}$ R = 5

16 N



+7,500
+750
+75
+ □
***
22,500
25,500
0
25,890
25,875
+ 15
345 R = 15
35) 25,890

20 N

9.2 Division #2

Exercise 1

Draw this table on the board.

Number of Bottles	Number of Bottles in Each Box	Number of Boxes Completely Filled	Number of Bottles Left Over
46	8		

Point to the appropriate places in the table as you say:

T: I would like to put 46 bottles into some boxes. Each box contains 8 bottles. How many boxes will I fill completely and how many bottles will be left over?

S: 5 boxes will be filled because 5 x 8 = 40.

S: 6 bottles will be left over.

Add this information to your table.

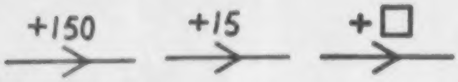
Number of Bottles	Number of Bottles in Each Box	Number of Boxes Completely Filled	Number of Bottles Left Over
46	8	5	6

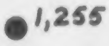
Continue in the same way for the information in the table below. Feel free to adjust the level of difficulty to the abilities of your class. Correct answers are in the boxes.

Number of Bottles	Number of Bottles in Each Box	Number of Boxes Completely Filled	Number of Bottles Left Over
46	8	5	6
184	10	$\boxed{18}$	$\boxed{4}$
1,256	10	$\boxed{125}$	$\boxed{6}$
1,256	100	$\boxed{12}$	$\boxed{56}$
5,268	100	$\boxed{52}$	$\boxed{68}$
486	20	$\boxed{24}$	$\boxed{6}$
486	40	$\boxed{12}$	$\boxed{6}$

Exercise 2

Put this information on the board.





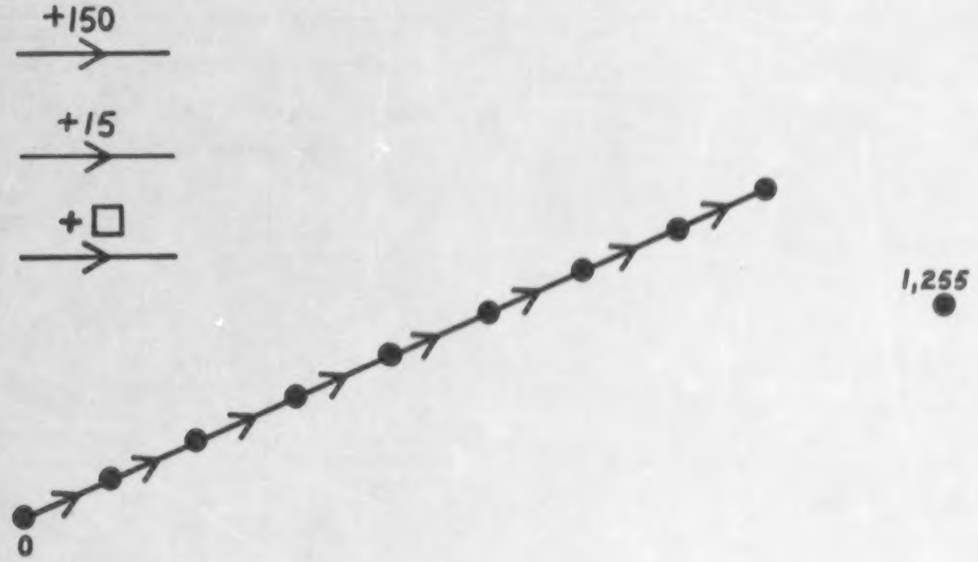
The black arrow should be drawn with white (or your regular) chalk.

T: Build a red-blue-white road from 0 to 1,255 using as few arrows as possible. Can you predict the number of blue arrows we should draw?

S: 8, because 8 x 150 = 1,200 and 9 x 150 = 1,350, which is too large.

20 N

Let the students have several minutes to work on this problem; then build the road collectively.



T (pointing to the last unlabeled dot): Which number is here? Why?

S: 1,200, because 8 x 150 = 1,200.

Label the last dot "1,200". If you find it necessary, the class could label all the unlabeled dots.

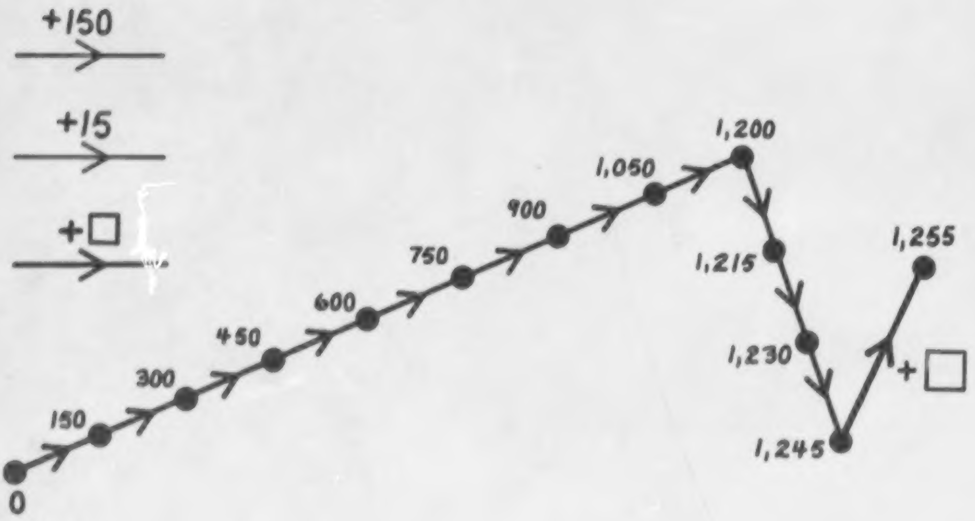
T: How far from 1,200 to 1,255?

S: 55.

T: How many red arrows should we draw?

S: 3 red arrows.

Add the three red arrows to your picture and ask a student to label the last dot. All the dots have been labeled in the picture below.



T: What is this white arrow (shown in black above) for?

S: +10.

Label the arrow "+10".

T: Bruce drew this same arrow road. However, he was using the road to solve a problem. What problem do you think Bruce was trying to solve?

Let the students express their opinions.

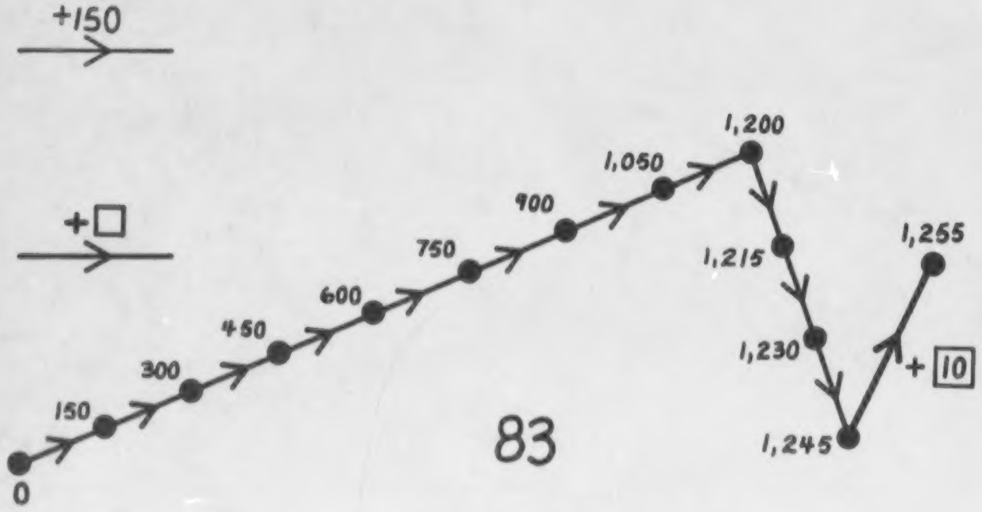
S: He might be putting 1,255 bottles into boxes of 15 bottles each. Each blue arrow could be for filling 10 boxes of 15 bottles each, and each red arrow could be for filling 1 box of 15 bottles.

20 N

T: How many boxes of 15 bottles each were completely filled?

S: 8 blue arrows for 80 boxes and 3 red arrows for 3 boxes — 83 boxes.

Write "83" with "8" in blue, and "3" in red near the arrow road.


+150
+□
0
150
300
450
600
750
900
1,050
1,200
1,215
1,230
1,245
1,255
+10
83

NOTE: Try to arrange your board space so that each sub-road continues in one direction (and different sub-roads in different directions). This arrangement highlights the link between the arrow road and the red and blue numerals.

T: How many bottles were left over?

S: 10 bottles.

Write this calculation on the board.

$$\begin{array}{r} 83 \\ 15\,\overline{)\,1{,}255} \end{array} \; R = 10$$

Exercise 3

Write this problem on the board.

$$24\,\overline{)\,7{,}949}$$

T: Let's use an arrow road to help solve this problem. How should we start?

S: We build the road from 0 to 7,949.

T: What kind of arrows could we use?

S: +24.

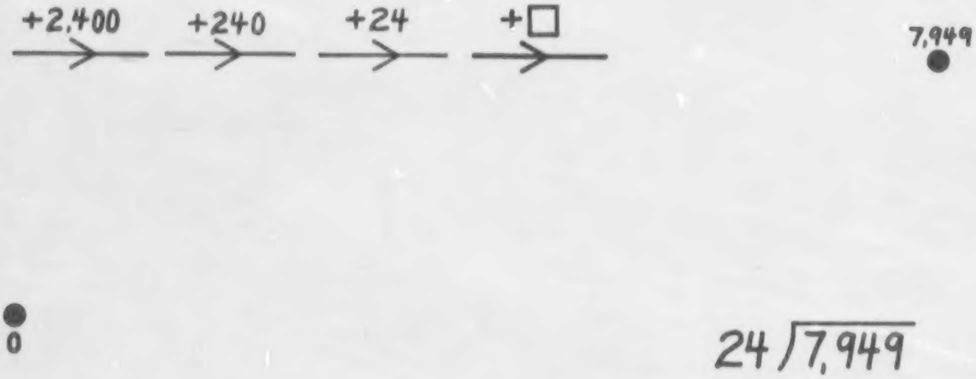
S: +240.

S: + 2,400. A + 2,400 arrow would be for filling 100 boxes of 24 bottles each.

S: We may need a "+☐" arrow for the remainder.

20 N

Put this information on the board.



+2,400
+240
+24
+□
7,949
0

$24\overline{)7,949}$

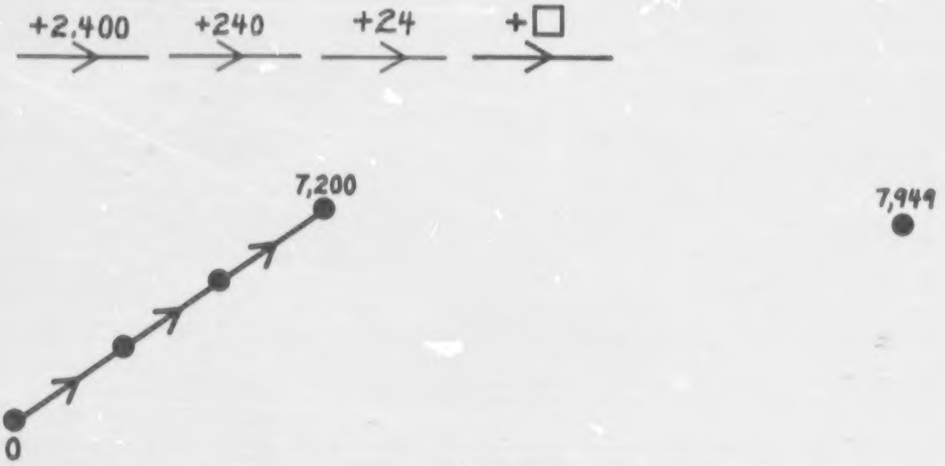
T: Which kind of arrow should we use first? Remember, we want to build the shortest road possible.

S: Start with the +2,400 arrow.

T: Can you predict how many +2,400 arrows we should use?

S: 3, because 3 x 2,400 = 7,200.

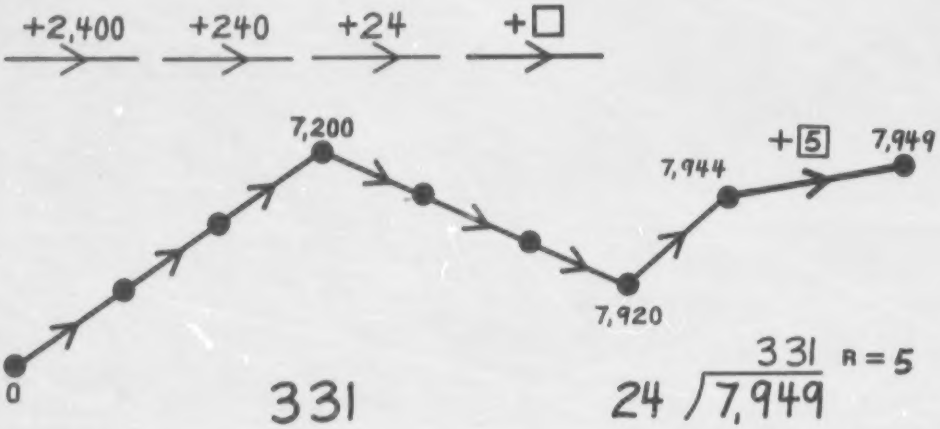
Draw the three arrows in your picture.



+2,400
+240
+24
+□
7,200
7,949
0

Continue in this manner until the road is completed. At some point, you may prefer to stop the collective discussion and ask the students to complete the problem independently.

The solution to this problem is given below.



331

$24\overline{)7{,}949}$ with quotient 331, R = 5

Write these problems on the board and ask the students to solve them.

$\boxed{*}$

$14\overline{)870}$ $15\overline{)732}$

$\boxed{**}$

$15\overline{)1{,}255}$ $45\overline{)2{,}387}$

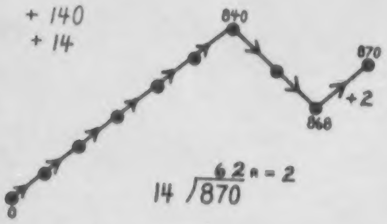
$\boxed{***}$

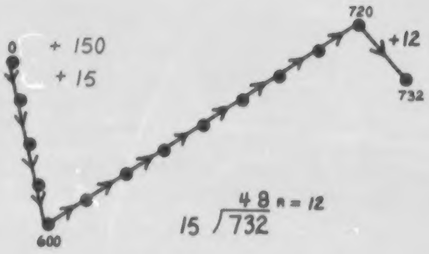
$55\overline{)18{,}673}$ $12\overline{)2{,}979}$

20 N

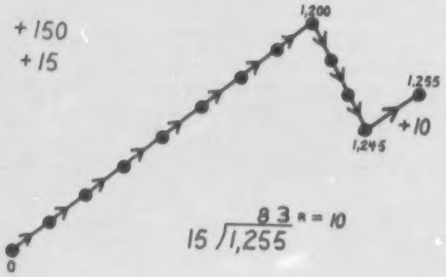
Solutions to these problems are given below.

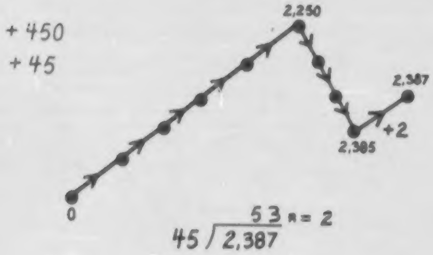
$\boxed{*}$


+ 140
+ 14
840
870
+2
868
0
62 R = 2
14 / 870

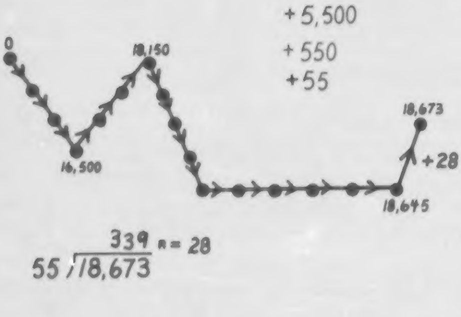

0
+ 150
+ 15
720
+12
732
600
48 R = 12
15 / 732

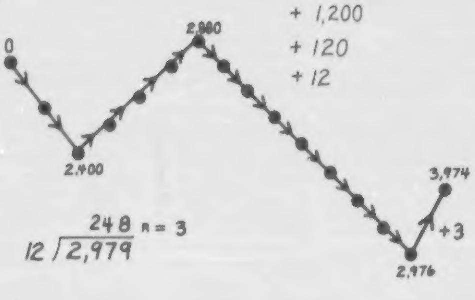
$\boxed{**}$


+ 150
+ 15
1,200
1,255
+10
1,245
0
83 R = 10
15 / 1,255


+ 450
+ 45
2,250
2,387
+2
2,385
0
53 R = 2
45 / 2,387

$\boxed{***}$

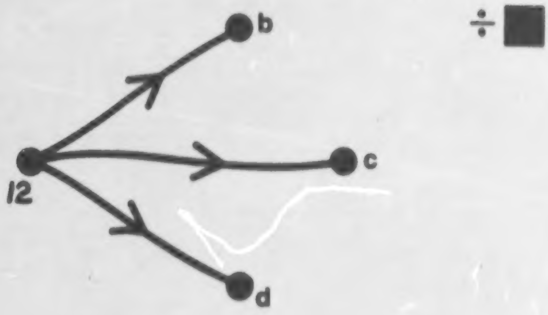

0
18,150
+ 5,500
+ 550
+ 55
18,673
16,500
+ 28
18,645
339 R = 28
55 / 18,673


0
2,880
+ 1,200
+ 120
+ 12
2,400
3,974
+3
2,976
248 R = 3
12 / 2,979

9.3 Division #3

Exercise 1

Draw this arrow picture on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: The red arrows are for division by some whole number.

Point to "12" and trace a red arrow ending at b.

T: 12 divided by some whole number equals this number (at b). Could this number be 3?

S: Yes, if the red arrow were for ÷ 4.

T: What other number could be here (at b)?

The class discussion might develop along these lines.

S: 4.

Label b "4".

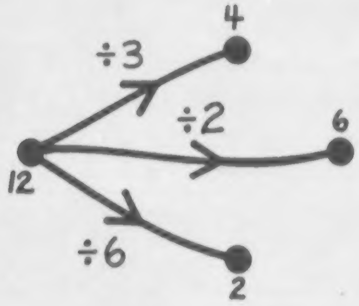
T: In order for "4" to go here, we have to find a whole number so that the red arrow is for divided by that whole number. What whole number will "work"?

27 N

S: 3.

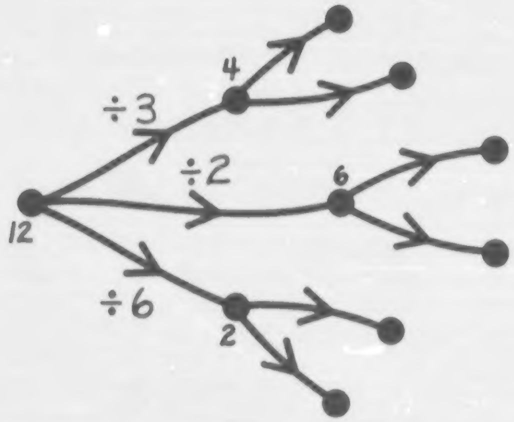
Label the arrow (starting at 12 and ending at 4) "÷ 3".

Continue in this manner for c and d. Each time labeling a dot, and then the corresponding arrow. A completely labeled arrow picture might look like this.



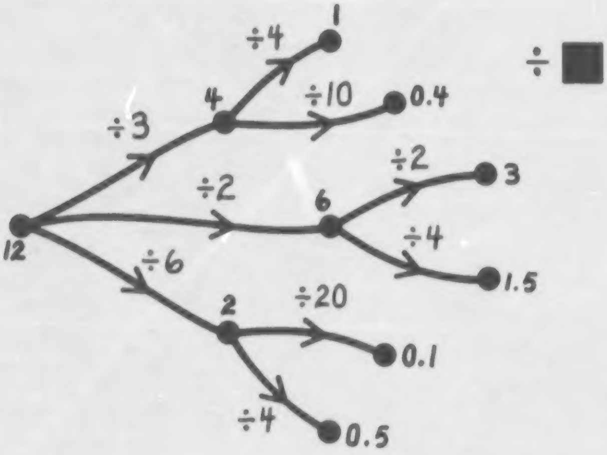
÷ ■

Extend your arrow picture with the addition of these arrows and continue to label dots and arrows.

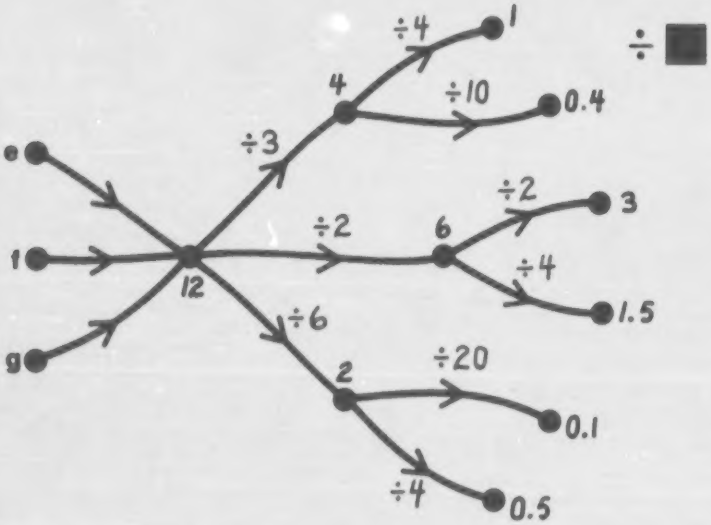


÷ ■

Your arrow picture might look like this upon completion of this activity.


÷ ■
1
÷4
4
÷10
0.4
÷3
÷2
3
÷2
6
÷4
12
÷6
1.5
2
÷20
0.1
÷4
0.5

Add these arrows ending at 12 to your picture.


÷ ■
1
÷4
4
÷10
0.4
÷3
e
÷2
3
÷2
6
f
12
÷4
1.5
÷6
g
2
÷20
0.1
÷4
0.5

Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

27 N

T (pointing to e): What number could be here?

S: 60.

Label e "60".

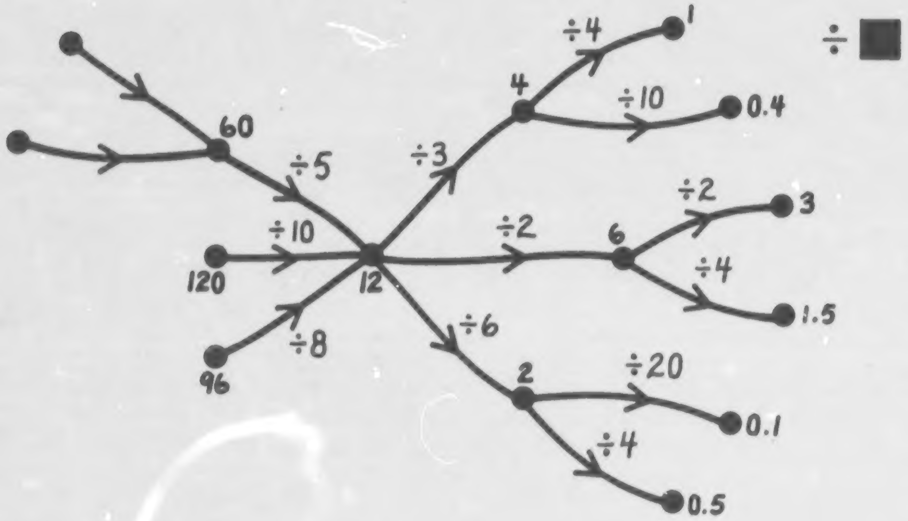
T (tracing the arrow starting at 60 and ending at 12): We must find a whole number so that this arrow is for divided by that whole number. What could this arrow be for?

S: ÷ 5.

Label the arrow "÷ 5".

Continue the activity in this manner, labeling f and g, and the respective arrows.

Expand your picture by adding these two arrows ending at 60 (or any one of the dots on the left).



Label the new dots and arrows in the same manner as above.

Draw a blue arrow starting from 12 and ending at 60.

T: What could this blue arrow be for? It is a return arrow for ÷ 5.

S: 5×.

T (pointing to "60", "120", and "96"): What kind of numbers are these?

S: Multiples of 12.

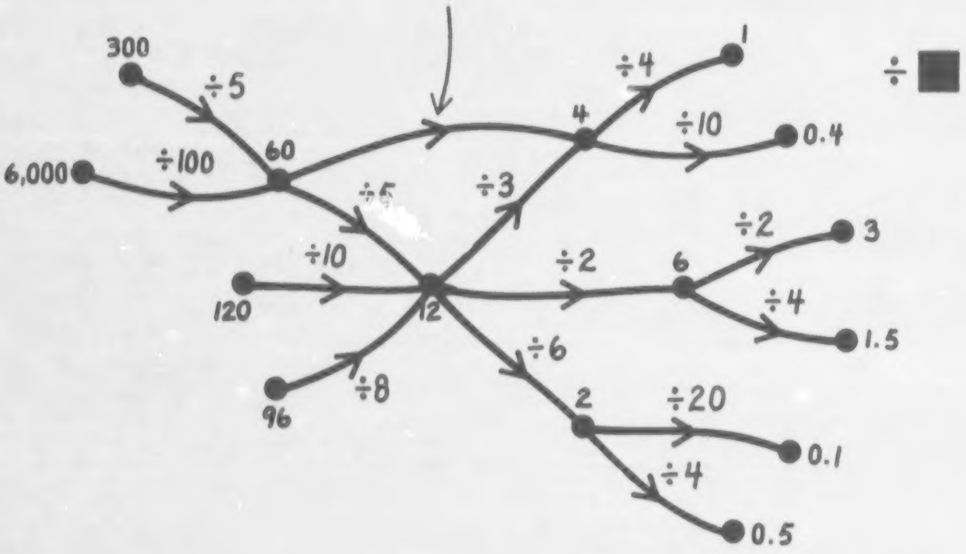
Other answers are correct, such as multiples of 2 or multiples of 6. Lead the discussion so that the class recognizes that they are indeed multiples of 12 and that is determined by the center dot being labeled "12".

T (pointing to "4", "6", and "2"): What kind of numbers are these?

S: Positive divisors of 12.

T: Do you see any other red arrows that we could draw?

Invite a student to trace an arrow; then you draw it. Suppose an arrow starting at 60 and ending at 4 is drawn as shown below.



27 N

T: This arrow must be for divided by some whole number. We can use the picture to find this whole number.

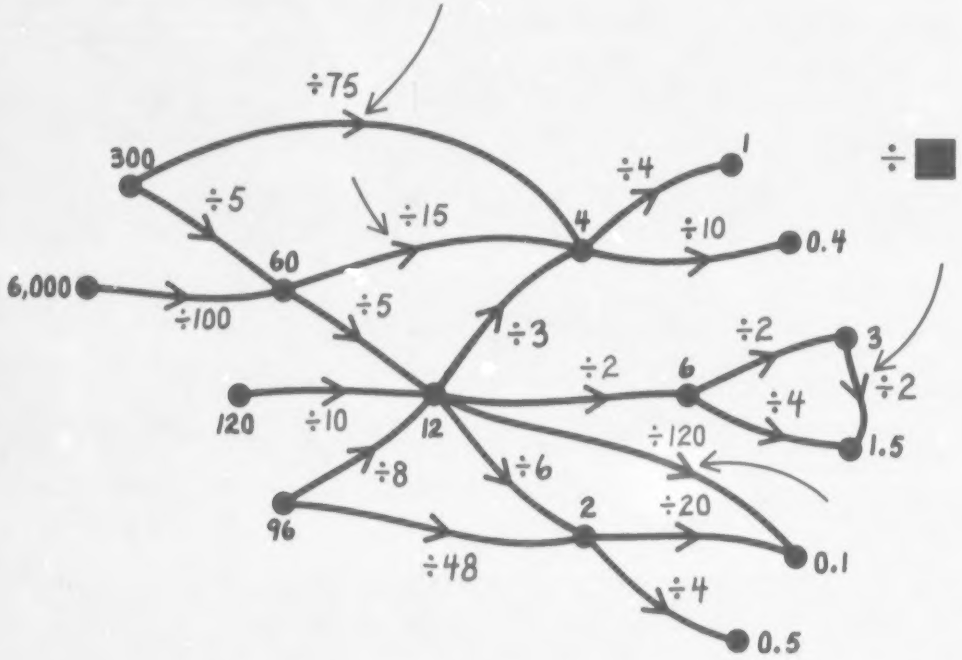
Start at 60 and trace the ÷ 5 arrow followed by the ÷ 3 arrow, saying:

T: ÷ 5 followed by ÷ 3.

S: ÷ 15.

Ask for a few more red arrows that can be drawn and label each of them divided by some whole number. Each time a red arrow is added, trace the appropriate arrows to stress that a label for the arrow can be found from a composition of two arrows.

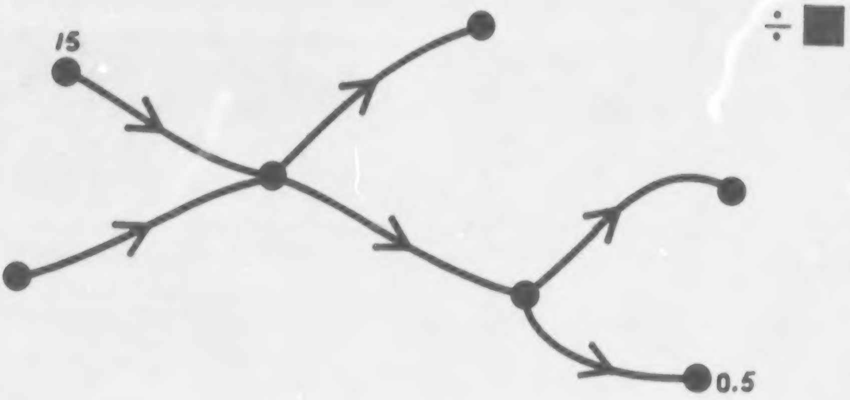
There are many possibilities; several are shown below.


÷75
300
÷5
÷15
4
÷4
1
÷10
0.4
÷ ■
6,000
÷100
60
÷5
÷3
÷2
3
÷2
6
÷2
÷4
1.5
120
÷10
12
÷120
÷8
÷6
96
2
÷20
0.1
÷48
÷4
0.5

379

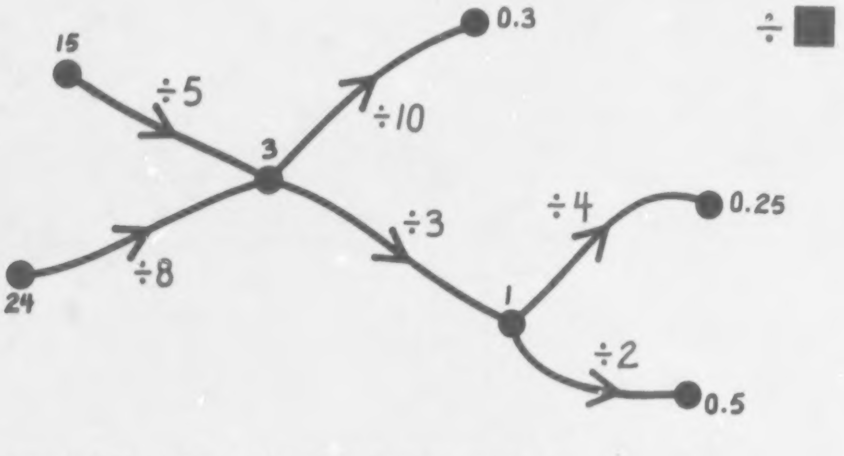
Exercise 2

Draw this arrow picture on the board.



T: Copy this picture on your paper. Label the dots and the arrows. There are many ways to complete this picture. Remember, each arrow should be labeled divided by some whole number.

Let the students have several minutes to work on this problem independently. Then ask several students to help you complete the picture on the board. One way to complete the picture is shown below.



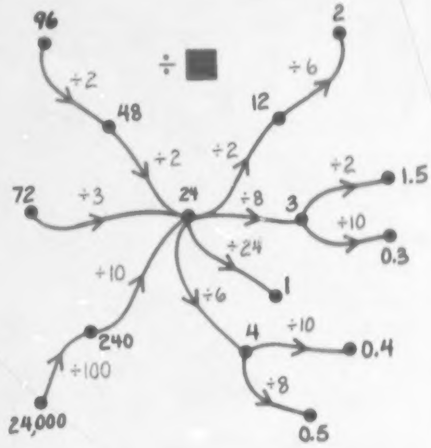
Worksheets N76, N77, N78, and N79 are available.

Name

N 76 *

" ÷ ■ " means each red arrow is for divided by some WHOLE number.

Label the dots. Label each arrow " ÷ ☐ ". Many solutions are possible.



96
÷2
48
÷ ■
2
÷6
12
÷2
÷2
72
÷3
24
÷8
3
÷2
1.5
÷10
0.3
÷24
1
÷10
240
÷100
24,000
÷6
4
÷10
0.4
÷8
0.5

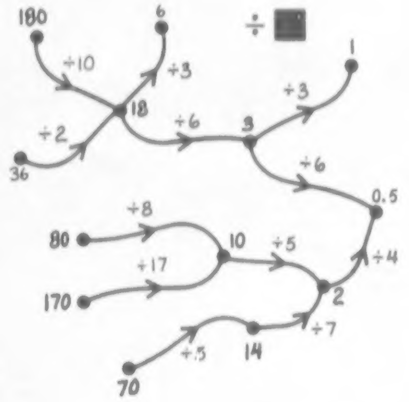
(Many other solutions are possible)

Name

N 77 **

" ÷ ■ " means each red arrow is for divided by some WHOLE number.

Label the dots. Label each arrow " ÷ ☐ ". Many solutions are possible.



180
÷10
6
÷3
÷ ■
1
18
÷2
36
÷6
3
÷3
÷6
0.5
÷8
80
10
÷5
÷17
170
÷4
2
÷7
÷5
14
70

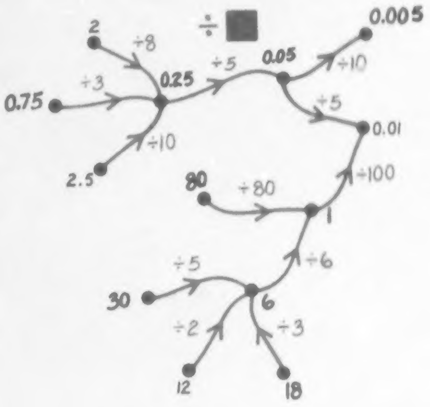
(Many other solutions are possible)

Name

N 78 ***

" ÷ ■ " means each red arrow is for divided by some WHOLE number.

Label the dots. Label each arrow " ÷ ☐ ". Many solutions are possible.



2
÷8
÷ ■
0.005
÷3
0.25
÷5
0.05
÷10
0.75
÷5
÷10
0.01
2.5
÷100
80
÷80
1
÷6
÷5
30
6
÷2
÷3
12
18

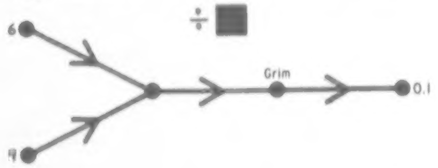
(Many other solutions are possible)

Name

N 79 ****

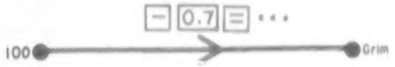
First clue

" ÷ ■ " means each red arrow is for divided by some WHOLE number.



Grim is 0.2 or 0.3 or 0.5 or 0.6 or 1 or 1.5.

Second clue



Grim is 0.6.

10

DETECTIVE STORIES

In this final chapter of the World of Numbers strand, many of the activities of the strand are brought together in the context of three detective stories.

CAPSULE LESSON SUMMARIES

Section 10.1: Solve a detective story in which two numbers are to be found. Make use of clues which involve the relation "is at least 20 smaller than", an arrow picture with arrows for $+100$ and $+80$, and the relation "is a positive divisor of"; and utilize the number line to examine these clues.

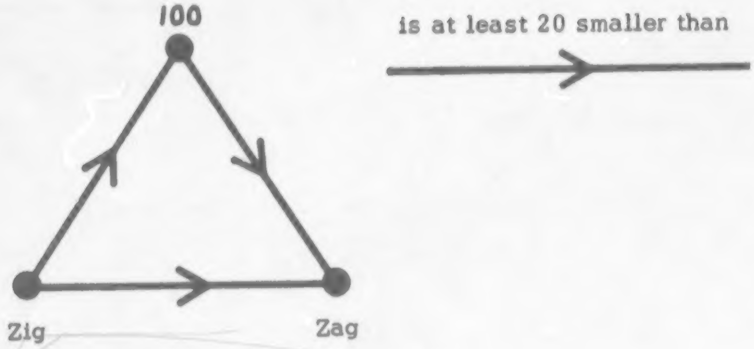
Section 10.2: Solve a detective story in which one number is to be found. Make use of clues which involve an arrow road with two arrows, one $\boxed{+}\boxed{3}\boxed{=}\ldots$ arrow and one $\boxed{+}\boxed{10}\boxed{=}\ldots$ arrow; the ones' boards of the Minicomputer with one weighted checker on it; and the relation $\boxed{-}\boxed{3}\boxed{=}\ldots$.

Section 10.3: Solve a detective story in which one number is to be found. Make use of clues which involve the relations $\boxed{+}\boxed{5}\boxed{=}\ldots$ and $\boxed{+}\boxed{4}\boxed{=}\ldots$, three Minicomputer boards with one weighted checker, and the relation $\boxed{-}\boxed{0.4}\boxed{=}\ldots$.

22 N

10.1 Who are Zig and Zag?

Draw this arrow picture on the board.



First clue

T: Zig and Zag are two secret numbers. They are in this arrow picture. Which numbers could Zig be? Why?

S: Zig could be 78, because 78 is 22 smaller than 100.

Ask the students for several more numbers that Zig could be. Accept as correct any number that is at least 20 smaller than 100; i.e., 80 or smaller than 80.

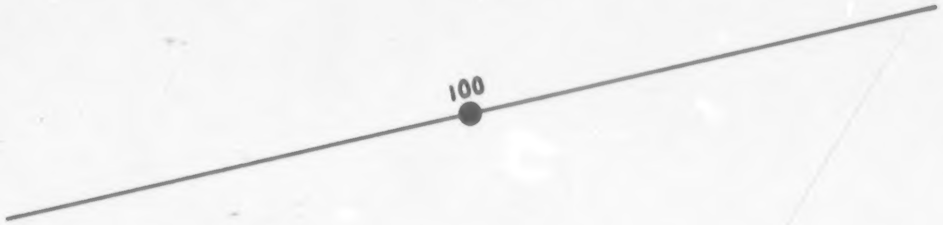
T: Which numbers could Zag be? Why?

S: Zag could be 125, because 100 is 25 smaller than 125.

Ask the students for several more numbers that Zag could be. Accept as correct any number that is at least 20 larger than 100; i.e., 120 or larger than 120.

Distribute Worksheet N80 and let the class work independently for a few minutes. Then in a collective discussion of the worksheet, ask students to explain their answers to the questions. Answers to the questions can be found at the end of this lesson.

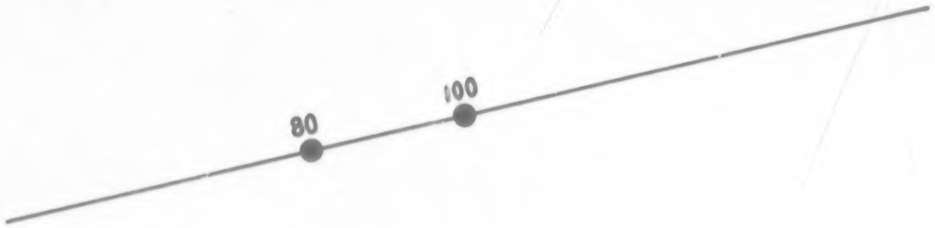
Draw this part of the number line on the board.



T: Which is the largest number Zig could be?

S: 80.

Put 80 on the number line.

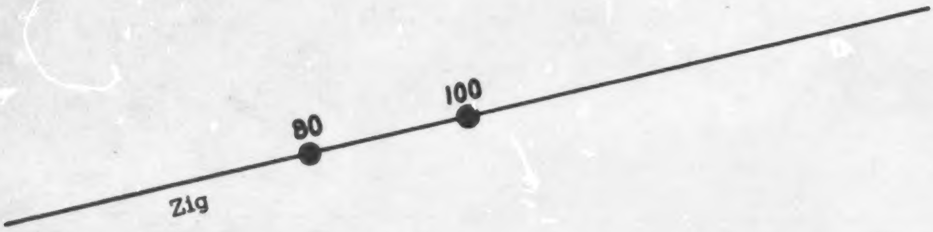


T: Where on the number line could Zig be?

S: Zig could be 80 or any number smaller (to the left) than 80.

22N

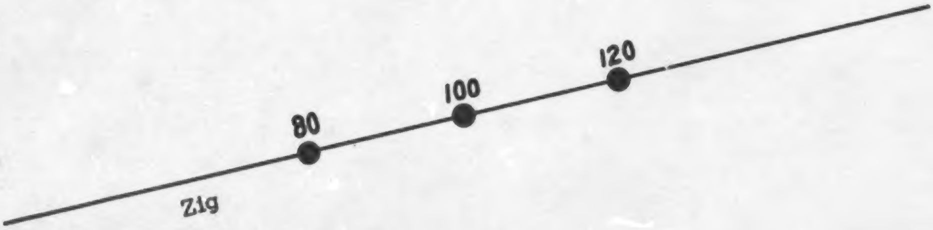
Color this part of the number line green.



T: Which is the smallest number Zag could be?

S: 120.

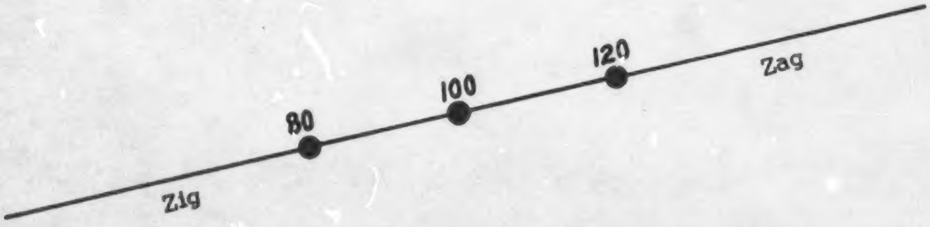
Put 120 on the number line.


120
100
80
Zig

T: Where on the number line could Zag be?

S: Zag could be 120 or any number larger (to the right) than 120.

Color this part of the number line green.

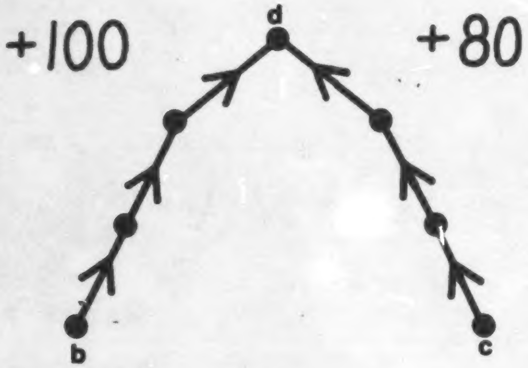

120
Zag
100
80
Zig

T: Which is the larger, Zig or Zag?

S: Zag.

Do not erase this part of the number line as it will be needed later.

Distribute Worksheet N81. Let the students independently label the dots for Zig and Zag. While the students are working, draw the arrow picture from the worksheet on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: Where is the largest number in this picture? Circle it on your worksheet.

Call on a student to come to the board and to point to d.

Second clue

T: Zig and Zag are the smallest two numbers in this picture. Write their names in the picture.

It is likely that most of the students will agree that b and c are the two smallest numbers in the arrow picture. However, there may be some disagreement over which of the two is Zig and which is Zag. In that case, ask:

T (pointing to b and d): How much smaller is this number (at b) than this number (at d)? Why?

S: 300 smaller. There are three + 100 arrows from one to the other.

T (pointing to c and d): How much smaller is this number (at c) than this number (at d)? Why?

S: 240 smaller, because 3 x 80 = 240.

T: Which one of these numbers is smaller?

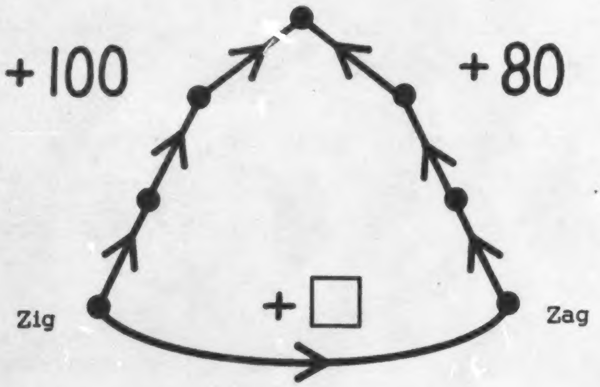
S: This number (at b) is smaller, because it is 300 smaller than this number (at d) while the other number (at c) is only 240 smaller than this number (at d).

S: Zig is smaller so it goes here (at b) and Zag goes here (at c).

A completed worksheet is shown at the end of this lesson.

T: The arrow picture gives us some more information about Zig and Zag. How much smaller is Zig than Zag? Write your answer on a sheet of paper.

If there are a few students who find this difficult, add the green arrow to your picture.



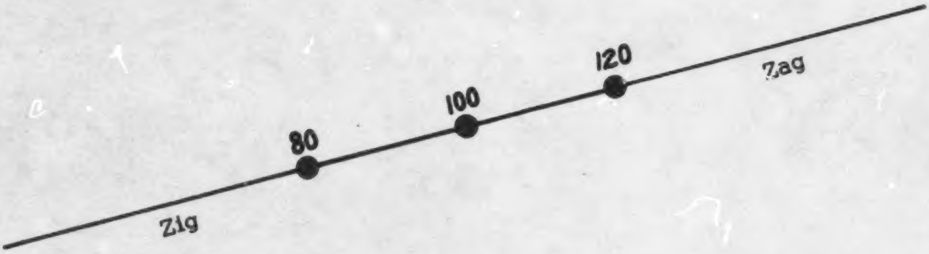
387

T: What could this arrow be for?

S: + 60, because 300 - 240 = 60.

S: Zig is 60 smaller than Zag.

Call the students' attention to the number line.



T: What are some numbers that Zag could be?

S: Zag could be 125.

T: If Zag were 125, what would Zig be?

S: 125 - 60 = 65. Zig would be 65.

S: Zag could be 150.

T: If Zag were 150, what would Zig be?

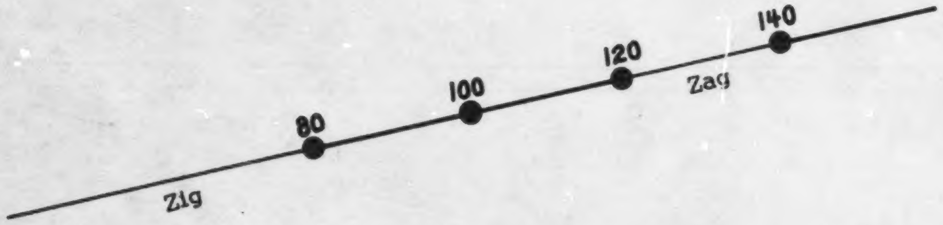
S: Zig would be 90, because 150 - 60 = 90. But Zig cannot be 90 so Zag cannot be 150.

Ask the students for several more examples of numbers that Zag could be.

T: Which is the largest number Zag could be? Why?

S: 80 is the largest number Zig can be and Zag is 60 larger, so 140 is the largest number Zag could be.

Put a dot for 140 on the number line and use your regular chalk to re-color the part of the number line to the right of 140.



T: Now we know that Zag could be 120 or 140 or a number between them. What are some numbers that Zig could be?

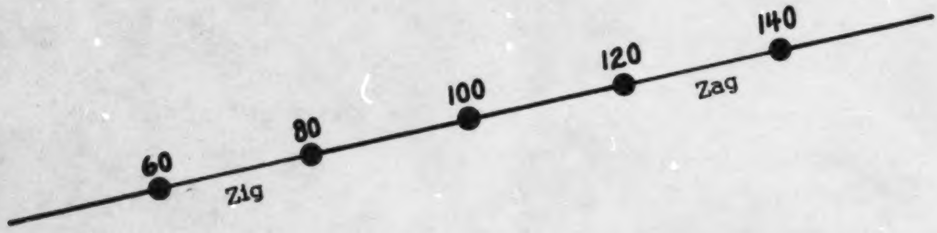
S: Zig could be 75, because 75 + 60 = 135 and Zag could be 135.

Ask for several more numbers that Zig could be.

T: What is the smallest number that Zig could be?

S: 120 is the smallest number that Zag could be and 120 - 60 = 60, so 60 is the smallest number Zig could be.

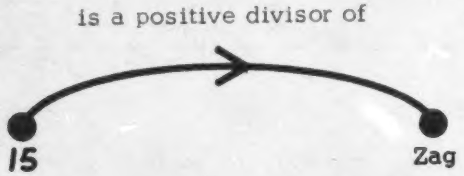
Put a dot for 60 on the number line and use your regular chalk to re-color the part of the number line to the left of 60.



T: Now we know that Zig could be 60 or 80 or any number between them.

Third clue

Draw this picture on the board.



T: Zag is in this arrow picture. What new information does this clue give us about Zag?

If there are a few students who are having difficulty with this clue, trace the return arrow starting at Zag and ending at 15 and ask:

T: What could this return arrow be for?

S: "Is a multiple of".

S: Zag is a multiple of 15.

Ask the students to give some examples of multiples of 15; e.g., $^{-}45$, 0, 15, 30, 60, 75, and so on.

T: Which multiples of 15 could Zag be?

S: 120 = 8 x 15.

S: 135 = 120 + 15.

Write this information on the board.

Zag is 120 or 135.
Zig is ___ or ___.

T: What do we know about Zig?

S: Zig is 60 smaller than Zag. Zig could be 60 or 75.

Write this information in the red box.

Zag is 120 or 135.
Zig is 60 or 75.

Fourth clue

T: Zig has exactly six positive divisors. Who is Zig and who is Zag?

Ask the students to name the positive divisors of 60 as you list them on the board (there are twelve), and to name the positive divisors of 75 (there are six) in a similar manner.

The class should conclude that Zig is 75 and Zag is 135.

Name ____________________ **N 80** *

Could Zig be 82? No

Could Zig be 75? Yes

Could Zig be 0? Yes

Could Zig be $^{-}5$? Yes

Could Zig be 80? Yes

Could Zag be 115? No

Could Zag be 120? Yes

Could Zag be 1,000,000? Yes

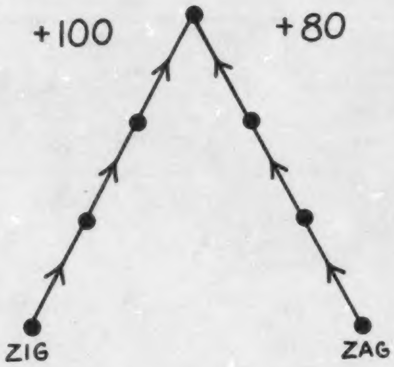
What is the largest number Zig could be? 80

What is the smallest number Zag could be? 120

Name ____________________ **N 81** *

Second clue

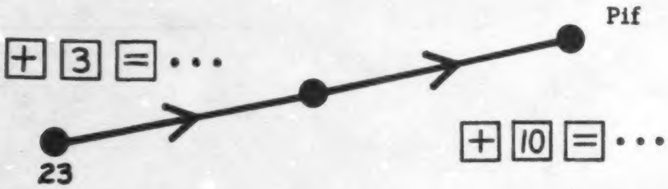
Zig and Zag are the two smallest numbers in this picture.



10.2 Who is Pif?

Distribute as many hand-calculators as you have available.

Draw this arrow picture on the board.



First clue

T: Pif is in this arrow picture. What are some numbers that Pif could be?

Let the students have a few minutes to explore this situation; then ask for some responses. (Answers: Pif could be 39; 48; 49; 85; 1,317; and so on.)

It might be necessary to remind the class that Pif is the ending number of the arrow road, rather than the middle number.

T: Which is the smallest number Pif could be?

S: 36.

T: Which is the next larger number after 36 that Pif could be?

S: 39.

T: How did you get 39?

S: I pressed $\boxed{+}\,\boxed{3}\,\boxed{=}\,\boxed{=}$ then $\boxed{+}\,\boxed{10}\,\boxed{=}$.

T: Find some more numbers that Pif could be. Try not to miss any numbers in the sequence.

Give the students a few minutes to work independently. Encourage everyone to find several of the numbers in the sequence.

T: Which is the next number?

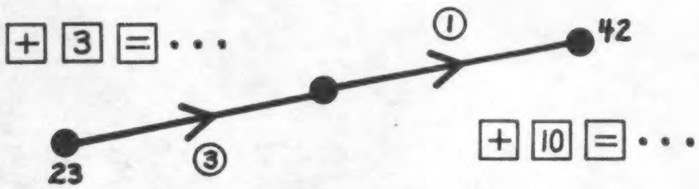
S: 42.

It will be helpful if you list in order on the board numbers that Pif could be. Continue by asking:

T: How did you get 42?

S: I pressed $\boxed{+}\,\boxed{3}\,\boxed{=}\,\boxed{=}\,\boxed{=}$ then $\boxed{+}\,\boxed{10}\,\boxed{=}$.

If you prefer, as a student is giving an explanation of this kind, write the information in the arrow picture as illustrated below.



29 N

It is not necessary to have an explanation of this kind for each number that Pif could be.

T: Next number?

S: 45 (press $\boxed{+}\ \boxed{3}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}$ and then $\boxed{+}\ \boxed{10}\ \boxed{=}$).

S: 46 (press $\boxed{+}\ \boxed{3}\ \boxed{=}$ and then $\boxed{+}\ \boxed{10}\ \boxed{=}\ \boxed{=}$).

S: 48.

S: 49 and 51.

Continue extending your list of numbers that Pif could be. Be sure to check any numbers that are disputed. When your list includes these numbers:

54, 55, 56, 57, 58, 59, 60, 61, 62, 63,

highlight them in some way (underline or bracket) and ask:

T: What do you think about this; ten numbers in a row?
What can we conclude about the list from here on?

S: We can get 64, 65, 66, 67, 68, and so on.

T: Do you think we will miss any whole numbers after 63?

S: No, all of the whole numbers larger than 63 will be there.

T: Why?

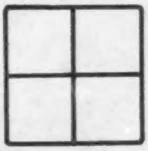
S: Start with 54 and add 10, that's 64. Do the same with 55, 56, 57, 58, 59, 60, 61, 62, and 63 and get 65, 66, and so on to 73. That's ten more numbers in a row. If 10 more is added to each of these numbers, the next ten numbers (from 74 to 83) are obtained. We can go on and on and get each number larger than 83.

Accept any reasonable explanation, even if it is not the same as the above. It may happen that a student will note that when three numbers in a row (54, 55, 56) appear, you can be sure that all numbers larger will appear. This is correct and an argument similar to the above can be made using $\boxed{+}\ \boxed{3}\ \boxed{=}$ ... instead of $\boxed{+}\ \boxed{10}\ \boxed{=}$ .... If a student suggests this explanation, praise the student, but it is not necessary to mention it yourself.

Second clue

Display one Minicomputer board and exactly one each of the new checkers:

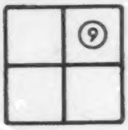
② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨



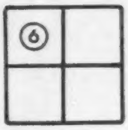
T: Now for the second clue:

Pif can be put on this Minicomputer board using exactly one of these new checkers: ② through ⑨.

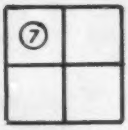
Ask a student to place one checker on the Minicomputer while the rest of the class tells you the number on the Minicomputer. The students should conclude that Pif could be one of the following numbers:



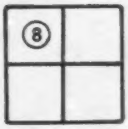
= 36



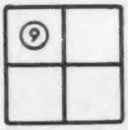
= 48



= 56



= 64



= 72

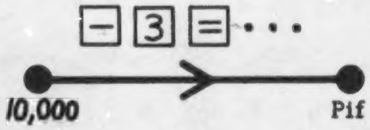
T: Pif could be 36 or 48 or 56 or 64 or 72. Are you sure there are no larger numbers that Pif could be?

S: Yes. A ⑨-checker on the 8-square is 72 and that is the largest number that can be put on the ones' board with exactly one new checker.

The class should conclude that Pif is 36 or 48 or 56 or 64 or 72.

Third clue

Draw this arrow picture on the board.



T: The third clue is that if you put 10,000 on the display of a hand-calculator and press $\boxed{-}\ \boxed{3}\ \boxed{=}$ . . . as many times as you like, Pif will appear on the display. Who is Pif?

Give the class a few minutes to explore this situation. If there are a few students who have difficulty with this clue, try asking:

T: If I press $\boxed{-}\ \boxed{3}$ and then $\boxed{=}$ 1,000 times, what number will appear on the display? Why?

S: 7,000 will appear, because 10,000 - 3,000 = 7,000.

T: Suppose I pressed $\boxed{=}$ 3,000 times instead. What number would appear?

S: 1,000, because 10,000 - 9,000 = 1,000.

T: We know that 1,000 will appear on the display. Now, do you know who is Pif?

The class should conclude that Pif is 64. Ask several students to explain how they found Pif. Two explanations from students in our development classes are given below.

S: I started from 1,000 and pressed $\boxed{-}$ $\boxed{3}$ $\boxed{=}$... until 964 appeared. Then I knew Pif had to be 64 because I could subtract 900 (press $\boxed{=}$ 300 more times) from 964.

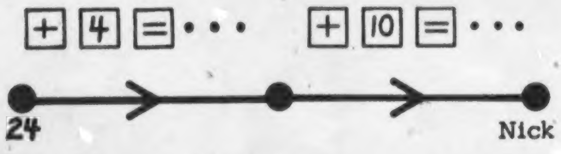
S: I started from 1,000 and subtracted 900. So I know 100 will appear on the display. Then I pressed $\boxed{-}$ $\boxed{3}$ $\boxed{=}$... until 64 appeared. If I keep pressing $\boxed{=}$, the other numbers (56, 48, and 36) do not appear on the display.

Accept any reasonable explanation along these lines.

Worksheets N82, N83, and N84 are available.

You may prefer to present Worksheet N82 collectively for those students who have difficulty getting started. The discussion of the first clue might develop in a manner similar to the following dialogue.

Draw this arrow picture on the board.



29 N

T: What are some numbers that Nick could be?

38, 42, 46, 48, 50, 52, 54, and so on are likely to be suggested. Encourage the students to find these numbers in order.

T: These are all even numbers. Do you think Nick could be an odd number?

S: No. We start with an even number and always add an even number.

T: Do you think we will miss any of the even numbers larger than 46?

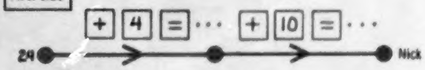
S: No. There are five in a row: 46, 48, 50, 52, and 54. Then if 10 is added to each, we get the next five (56, 58, 60, 62, 64) and so on. We can display every even number larger than 46.

Accept any reasonable explanation, even it it is not like the above.

If necessary, the second and third clues could be discussed collectively, too. In any case, the students should conclude that Nick is 64.

Name ______________________ N 82 *

First clue



Nick could be 38 ; 42 ; 46 ; 48 ; 50 ; 52 ; 54 ; 56 ;
58 ; 60 ; 62 ; 64 ; 66 ; 68 ; 70 ; 72 ; and so on.

Second clue

Nick can be put on this Minicomputer board using exactly one of these checkers:

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

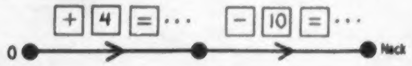
Nick is 48 or 56 or 64 or 72 .

Third clue

Nick is a square number. Nick is 64 .

Name ______________________ N 83 **

First clue

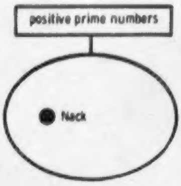


Nack could be $^{-}6$; $^{-}2$; 2 ; 6 ; 10 ; 14 ; 18 ; 4 ;
8 ; 22 ; 12 ; $^{-}4$; $^{-}8$; 26 ; 16 ; 30 ; and so on.

What do you notice about the numbers that Nack could be?

Nack can be any even number except 0
(since 0 is already in the picture).

Second clue

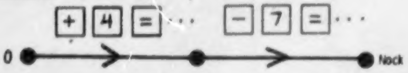


positive prime numbers
Nack

Nack is 2 .

Name ______________________ N 84 ***

First clue



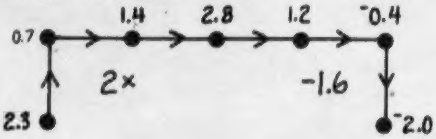
Nock could be $^{-}3$; $^{-}10$; $^{-}17$; 1 ; $^{-}6$; $^{-}13$; 5 ; $^{-}2$;
$^{-}9$; 13 ; 6 ; $^{-}1$; 17 ; 10 ; 3 ; 21 ; and so on.

What do you notice about the numbers that Nock could be?

Nock is any integer except 0 (since
0 is already in the picture).

Second clue

Nock is in this arrow picture.



0.7
1.4
2.8
1.2
⁻0.4
2×
−1.6
2.3
⁻2.0

Nock is $^{-}2$.

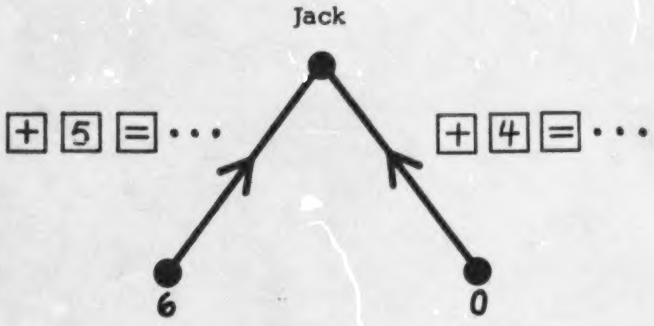
34 N

10.3 Who is Jack?

Distribute as many hand-calculators as you have available.

First clue

Draw this arrow picture on the board.



T: Jack is in this arrow picture. What are some numbers that Jack could be? You may use your hand-calculator to help you.

Let the class have a few minutes to explore this situation; then ask students to suggest numbers that Jack could be; for example, 16, 36, 56, 76, 96, and so on.

T: Which is the smallest number Jack could be?

S: 16.

T: The next larger number Jack could be?

S: 36.

List the numbers Jack could be on the board and continue in this manner until it is obvious that most students recognize a pattern.

Your list might look like this:

16, 36, 56, 76, 96, 116, 136, 156, ...

T: Do you notice anything interesting about this list?

S: Each number is 20 larger than the preceding one.

S: The tens' digit is odd and the ones' digit is always 6.

T: Which is the largest number smaller than 1,000 that Jack could be? Write it on your paper.

S: 996.

T: Which is the smallest number larger than 1,440 that Jack could be?

S: 1,456; I added 1,400 to 56.

T: Which is the largest number smaller than 10,000 that Jack could be?

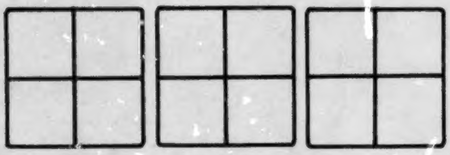
S: 9,996.

34 N

Second clue

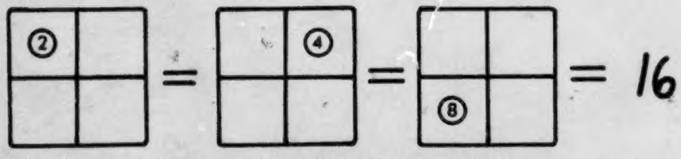
Display three Minicomputer boards and this set of new checkers:

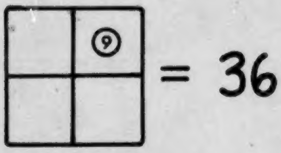
② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

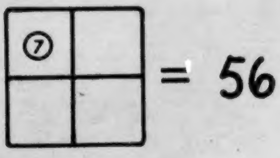


T: Jack can be put on the Minicomputer using exactly one of these checkers. Show me some numbers that Jack could be.

Ask a student to place one checker on the Minicomputer while the rest of the class tells you the number on the Minicomputer. The class should find that Jack could be one of these numbers:







T: Do you think these numbers are the only three numbers that Jack could be? Why?

Let the students consider this situation for a moment.

S: 72 is the largest number that can be put on the ones' board of the Minicomputer using exactly one of the new checkers. Any larger number would have to be put on the tens' board or the hundreds' board.

T: What do you know about the numbers that can be put on the tens' board or the hundreds' board with exactly one new checker?

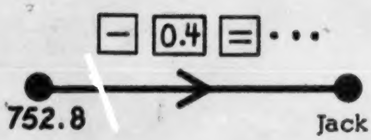
S: They all end in 0.

T: What about the numbers Jack could be?

S: They end in 6; so 16, 36, and 56 are the only numbers Jack could be.

Third clue

Draw this arrow picture on the board.



T: The third clue is that if you put 752.8 on the display of a hand-calculator and press $\boxed{-}$ $\boxed{0.4}$ $\boxed{=}$ . . . as many times as you like, Jack will appear on the display. Who is Jack? You may use your calculators for help.

34 N

Let the students have a few minutes to explore this situation independently. For those students who have difficulty working with this clue, suggest:

T: Display 752.8 on your calculators. Hide the display, and press $\boxed{-}$ $\boxed{0.4}$ $\boxed{=}$ $\boxed{=}$. What number should be on the display?

S: 752.

To highlight the emerging pattern, write "752" on the board and continue listing other whole numbers that appear. These numbers are suggested in the following dialogue.

T: Hide the display again and press $\boxed{=}$ five more times. What number should be on the display?

S: 750.

T: Hide the display. Press $\boxed{=}$ until you get the next smaller whole number on the display. Which number is it? How many times did you press $\boxed{=}$?

S: 748. I pressed $\boxed{=}$ five times.

T: What is the next smaller whole number to appear on the display?

S: 746.

Continue in this manner until most of the students are aware of a pattern.

T: Do you notice anything interesting about this list?

S: Each number is 2 less than the preceding number.

T: Why?

S: 5 x 0.4 = 2.

T: Who is Jack?

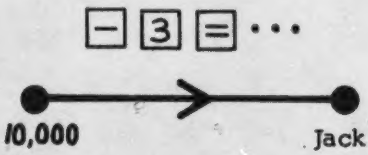
S: Jack could be 16 or 36 or 56.

T: Is this a very helpful clue?

S: No; we still have the same possibilities for Jack.

Fourth clue

Draw this arrow picture on the board.



T: Jack is in this arrow picture. Who is Jack?

Let the students have a few minutes to work on this problem independently.

S: Jack is 16.

T: How did you get 16?

S: I know I can get 1,000 (10,000 - 9,000 = 1,000) and I know I can get 100. From 100 I can get 16, but not 36 or 56.

S: I started with 10,000 and did not get 9,956 or 9,936; but I did get 9,916. So Jack must be 16 because I just subtract 9,900.

34 N

Another way of looking at this clue is described below. If this idea is not mentioned, suggest it yourself.

T: Think about 10,000. Is it a multiple of 3?

S: No, it is a multiple of 3 plus 1 (10,000 = 9,999 + 1).

T: Which of the three numbers that Jack could be is also a multiple of 3 plus 1?

S: 16 = 15 + 1. 10,000 and 16 are both multiples of 3 plus 1.

If there is time remaining, return the students' copies of A Variety of Problems 6 and let them spend the rest of the period working independently.